MATH 7230 Homework 2 - Spring 2017

Due Thursday, Feb. 1 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

Special note: You may turn in Problems 6 – 9 from Homework 1 as part of this assignment.

1. Recall that on Homework 1 Problem 8 you showed that $\sum_{n\geq 1} \frac{1}{n}$ diverges. In this problem you will use this fact to prove that $\sum_{p \text{ prime}} \frac{1}{p}$ also diverges (incidentally giving yet another proof of the infinitude of the primes).

   (a) Show that if $0 < x < 1$, then $1 + x < \dfrac{1}{1-x} < 1 + 2x$.

   (b) Show that if $x > 0$, then $1 + x < e^x$.

   (c) Use parts (a) and (b) to show that

   $$\prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p}} < \exp\left(2 \sum_{p \text{ prime}} \frac{1}{p}\right).$$

   (d) Conclude that $\sum_{p \text{ prime}} \frac{1}{p}$ must diverge.

2. In lecture we saw that the expression $n^{\frac{1}{\log\log n}}$ is "sub-polynomial" in the sense that it is smaller than any power of n. In this problem you will show that it is "super-logarithmic" in the sense that it is larger than any power of the logarithm.

   (a) As a warm-up, prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

   (b) What can you say about $\lim_{n\to\infty} n^{\frac{1}{\log(n)}}$?
   Hint: Is this a trick question?

   (c) Suppose that $k > 0$. Prove that

   $$n^{\frac{1}{\log\log n}} \gg (\log n)^k.$$

The main purpose of Problems 3 – 4 is to show a result that can be thought of as an extremely weak version of Goldbach's Conjecture: that every integer can be written as the sum of two squarefree numbers.

There are a number of intermediate results regarding the sum of two integers with a small number of factors. The strongest is due to Chen (1973), and states that any even integer is the sum of a prime and an "almost prime" of order 2: $n = p_1 + p_2$ or $n = p_1 + p_2 p_3$ where all p_i are prime.

3. MV 2.1.2. You will need to use a pigeonhole argument in part (d).
4. MV 2.1.4.

As discussed in lecture, the most naive quantified version of the Twin Prime Conjecture uses Cramer's model (i.e. that n is randomly prime with probability $\frac{1}{\log(n)}$) to estimate that the number of Twin Primes up to X is asymptotically $\frac{X}{(\log(X))^2}$. However, this is believed to be incorrect, as it ignores all "local" obstructions (e.g. reducing modulo 2 shows that n and $n+1$ can't both be prime).

A refined conjecture instead has $\Pi_2 \frac{X}{(\log(X))^2}$, where Π_2 is the *Twin Prime Constant.* This is calculated by using "singular series" for the local obstructions, and is given by

$$\Pi_2 := \prod_{\substack{p \text{ prime} \\ p>2}} \left(1 - \frac{1}{(p-1)^2}\right) = 0.660162\ldots.$$

In this problem you will derive this constant.

For this problem you should assume (though we will soon prove) the Prime Number Theorem in arithmetic progressions (PNT-AP), which states that if $(r, m) = 1$, then

$$\begin{aligned}\pi_{r,m}(X) &:= \#\{n \le X \mid n \text{ prime and } n \equiv r \pmod m\} \\ &\sim \frac{1}{\phi(m)}\pi(X) \sim \frac{1}{\phi(m)} \frac{X}{\log X},\end{aligned}$$

where ϕ is Euler's *totient* function.

5. (a) Show that when considering $n \le X$, it is a reasonable approximation to replace $\log(n)$ by $\log(X)$. Note that this is not quite true for all n; for example, if $n = \sqrt{X}$, then $\log(n) = \frac{1}{2}\log(X)$. However, show that as $X \to \infty$, the proportion of such n approaches 100%; i.e., for any $\varepsilon > 0$, show that

$$\lim_{X\to\infty} \frac{1}{X} \cdot \#\left\{n \le X \;\middle|\; \left|\frac{\log(n)}{\log(X)} - 1\right| < \varepsilon\right\} = 1.$$

Remark: Depending on the precise application, this property can be quantified in other ways; in fact, the statement above is not quite what is needed for counting twin primes.

(b) Let $P := p_1 p_2 \cdots p_r$ be the product of the first r primes for some fixed (for now) r. Suppose that r is relatively prime to P. Apply PNT-AP to conclude that the proportion of primes in this residue class is approximately

$$\frac{\#\{n \le X \mid n \text{ prime and } n \equiv r \pmod P\}}{\#\{n \le X \mid n \equiv r \pmod P\}} \sim \frac{P}{\phi(P)\log(X)}.$$

(c) Now consider all possible Twin Prime pairs $(n, n+2)$ with $n \le X$. First, show that the probability that both n and $n+2$ are relatively prime to P is $\prod_{p|P} \frac{p-2}{p}$.

Then assume that both n and $n+2$ are prime according to the probability from part (b) to conclude that the total proportion of Twin Prime pairs is

$$\prod_{p|P} \frac{p(p-2)}{(p-1)^2} \frac{1}{(\log(X))^2}.$$

The final statement is obtained by letting $r \to \infty$ (though this must happen much, much slower than $X \to \infty$).