MATH 7230 Homework 3 - Spring 2017

Due Thursday, Feb. 8 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

1. MV 1.1.3. This problem is an interesting example of using basic complex analysis to prove a result about integer congruences!

2. MV 1.1.7. This problem illustrates an application where generating functions are used to find a very simple answer to a seemingly complicated counting problem. Verify the result for a few small cases, say $p = 2, 3$ and $k = 1, 2$: write out all the polynomials and their factorizations.

3. In this problem you will prove the discrete version of Abel's "partial summation". Suppose that $f, g : \mathbb{N} \to \mathbb{C}$ are two arithmetic functions, with corresponding accumulation functions
$$F(n) := \sum_{1 \leq m \leq n} f(m), \qquad G(n) := \sum_{1 \leq m \leq n} g(m).$$
Prove that
$$\sum_{1 \leq n \leq N} f(n)G(n) = F(N)G(N) - \sum_{1 \leq n \leq N} F(n-1)g(n).$$
Remark: This is analogous to integration by parts using the finite difference operator $\Delta(h(n)) := h(n) - h(n-1)$ *instead of the derivative; then* $F(n)$ *is the "anti-difference" of* $f(n)$.

4. One of the basic applications of partial summation is Abel's Lemma for series: Suppose that $\{a_n\}_{n \geq 1}$ is a monotone (decreasing or increasing) sequence, and $\{b_n\}$ is a sequence such that $\sum_{n \geq 1} b_n$ converges. Then $\sum_{n \geq 1} a_n b_n$ converges.
   (a) One key step in the proof is to apply the finite-difference operator to the sequence of a_n. Define $a'_n := a_n - a_{n-1}$, so that (setting $a_0 := 0$ for convenience)
   $$a_n = a'_n + a'_{n-1} + \cdots + a'_1.$$
   Now apply the discrete partial summation from Problem 3.
   (b) Complete the proof by bounding each term absolutely. You will need to use the fact that all a'_n are the same sign (why?).

5. In lecture we discussed Chebyshev's "weighted" prime counting function, $\theta(X) := \sum_{p \leq X} \log(p)$, and used partial summation to prove an implication of the Prime Number Theorem (PNT):
$$\pi(X) \sim \frac{X}{\log(X)} \qquad \Longrightarrow \qquad \theta(X) \sim X.$$

Prove the converse implication (which shows that Cheyshev's condition is equivalent to PNT).

Hint: Apply partial summation to $\pi(X) = \sum_{p \le X} 1 = \sum_{n \le X} \mathbf{1}_{\mathcal{P}}(n) \log(n) \cdot \frac{1}{\log(n)}$*, where* $\mathbf{1}_{\mathcal{P}}$ *is the indicator function for the set of primes* $\mathcal{P}$*.*

In lecture we used Euler's summation by parts in order to prove Stirling's formula up to a constant; in particular, we found that

$$n! \sim c'\sqrt{n}\left(\frac{n}{e}\right)^n. \tag{1}$$

In Problems 6–7 you will learn two different proofs that $c' = \sqrt{2\pi}$.

6. Euler stated that following product expansion for the sine function:

$$\sin(x) = x \prod_{n \ge 1}\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

   Euler's roughly used the Fundamental Theorem of Algebra (which is only true for polynomials!) to argue that since $\sin(x)$ has a zero at $n\pi$ for any $n \in \mathbb{Z}$, there must be a linear factor $(x - n\pi)$ in the product. Furthermore, since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, the product must be scaled by a constant such that $\sin(x) = x + \cdots$. This argument is not legal (why?), but Euler's expansion is true, as was proven later by Weierstrass.

   (a) Prove that if x is fixed, then for large $N \to \infty$ the series may be truncated to obtain an approximation; i.e. that

$$\sin(x) \sim x \prod_{n=1}^{N}\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

   (b) Now plug in $x = \frac{\pi}{2}$ to obtain Wallis' formula (fill in the details of the calculation):

$$1 \sim \frac{\pi(2n)!(2n+1)!}{2^{4n+1}(n!)^4}. \tag{2}$$

   (c) Finally, plug in (1) for each factorial and solve for c'.

7. In this problem you will prove Wallis' formula (2) using a family of trigonometric integrals. Let

$$I_n := \int_0^{\frac{\pi}{2}} \sin^n(x)dx.$$

   (a) Evaluate I_0 and I_1.

   (b) Use integration by parts to prove that for $n \ge 2$,

$$I_n = \frac{n-1}{n} I_{n-2}.$$

   Conclude that $I_n \sim I_{n-2}$ as $n \to \infty$.

   (c) Prove that $I_{2n+2} < I_{2n+1} < I_{2n}$. Conclude that $I_{2n+1} \sim I_{2n}$ as $n \to \infty$.

   (d) Now write I_{2n+1} and I_{2n} in terms of factorials to derive Wallis' formula.