MATH 7230 Homework 4 - Spring 2017

Due Thursday, Feb. 22 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

Problems 1–2 address additional aspects of Chebyshev's bounds.

1. Chebyshev's Theorem gives the asymptotic order for the von Mangoldt accumulation function, $\psi(X) = \sum_{n \leq X} \Lambda(n)$; in particular, that there are constants c_i such that if X is sufficiently large,
$$c_1 X + o(X) \leq \psi(X) \leq c_2 X + o(X). \tag{1}$$
In lecture we achieved $c_1 = \log(2) = 0.69\ldots$ and $c_2 = 2c_1 = 1.38\ldots$ by considering the function $A_2(X) := T(X) - 2T\left(\frac{X}{2}\right)$ (see below). In this problem you will improve on these constants, getting closer to the true asymptotic main term of $\psi(X) \sim X$ (which is equivalent to the Prime Number Theorem).

   (a) The logarithmic accumulation function is $T(X) := \sum_{n \leq X} \log(n)$; we showed in lecture that $T(X) = \sum_{d \leq X} \psi\left(\frac{X}{d}\right)$. Now consider
$$A_3(X) := T(X) - T\left(\frac{X}{2}\right) - 2T\left(\frac{X}{3}\right) + T\left(\frac{X}{6}\right),$$
and write $A(X)$ in terms of $\psi(X)$ in as simple an expression as possible.

   (b) Using properties of alternating series, show that
$$\psi(X) - \psi\left(\frac{X}{3}\right) \leq A(X) \leq \psi(X),$$
with strict inequalities for sufficiently large X. Use this to show that
$$A(X) \leq \psi(X) \leq A(X) + A\left(\frac{X}{3}\right) + A\left(\frac{X}{9}\right) + \cdots.$$

   (c) Recall that Stirling's formula gives the following asymptotic terms for the logarithmic accumulation function,
$$T(X) := \sum_{n \leq X} \log(n) = X \log X - X + O(\log X).$$
Plug in to part (b) to conclude that
$$c_1 X + O(\log X) \leq \psi(X) \leq \frac{3}{2} c_1 X + O\left((\log X)^2\right),$$

where $c_1 = 0.78\ldots$. You should find an exact formula for c_1; the decimal approximation is given so you can check your work!

Remark: In order to convert this to an exact statement for large X, it is necessary to shift the constants: If $\varepsilon > 0$, then for sufficiently large X

$$(c_1 - \varepsilon)X \leq \psi(X) \leq \left(\frac{3}{2}c_1 + \varepsilon\right) X.$$

2. This problem is a somewhat open-ended (and possibly quite time-consuming – be forewarned!) continuation of Problem 1. Note that there is a logic behind the subscripts in the A_j; try to understand it!

   (a) Chebyshev's original proof used

   $$A_6(X) := T(X) - T\left(\frac{X}{2}\right) - T\left(\frac{X}{3}\right) - T\left(\frac{X}{5}\right) + T\left(\frac{X}{30}\right).$$

   Show that this gives (1) with

   $$c_1 = \frac{7}{15}\log 2 + \frac{3}{10}\log 3 + \frac{1}{6}\log 5, \qquad \text{and} \qquad c_2 = \frac{6}{5}c_1.$$

   (b) There is at least one additional intermediate linear combination that gives better constants that A_2 and A_3 (but worse than A_6). In particular, there is a relatively simple choice of $A_4(X)$ such that $c_2 = \frac{4}{3}c_1$. Find this function and calculate the constants.

   (c) In lecture I incorrectly stated that the same arguments could be applied to $T(X) - 3T\left(\frac{X}{3}\right)$. However, this is wrong – explain why! Can you modify the arguments to work in this case (i.e., even though the series are not term-by-term alternating)? Please share any ideas that you have, as this is not well-understood.

Remark: Diamond and Erdös (1981) showed that Chebyshev's ideas can be adapted to obtain c_1 and c_2 that are arbitrarily close to 1, but this is still weaker than the Prime Number Theorem.

Note that (1) implies that $\psi(2X) - \psi(X) = aX + o(X)$ for some positive $a > 0$. This means that for sufficiently large X, there must be at least one prime between X and $2X$. By taking a bit more care with the lower-order terms, Chebyshev proved *Bertrand's postulate:* For **any** $n \geq 1$, there is a prime satisfying $n < p \leq 2n$.

In Problems 3–4 you will work through an elementary proof of Bertrand's postulate due to Erdös.

3. (a) Show that

   $$2^{2n} \leq (2n+1)\binom{2n}{n}.$$

   The expression $\binom{2n}{n}$ is known as a "central Binomial coefficient", since it occurs in the central column of Pascal's triangle.

   Hint: Apply the Binomial Theorem to $(1+1)^{2n}$, and bound by the largest term in the sum.

   (b) Prove that if $p^r \mid \binom{2n}{n}$, then $p^r \leq 2n$; in other words, the central Binomial coefficients are not divisible by "large" prime powers.

   Hint: Count the multiplicity of the powers of p in the numerators $(2n)!$, and compare to the denominator $(n!)^2$.

(c) Note that if $2 < n < p \le 2n$, then $p \mid \binom{2n}{n}$ exactly once. Now consider $\frac{2n}{3} < p \le n$, and show that such a prime does **not** divide $\binom{2n}{n}$.

4. (a) Prove that $\prod_{p \le n} p \le 4^n$. This product is sometimes referred to as "n-primorial", which is denoted by $n\#$. However, we have also seen this in the context of Chebyshev's accumulation functions; note that

$$\prod_{p \le X} p = e^{\vartheta(X)},$$

so $\psi(n) = \log(n\#)$ (this is MV 2.2.1(a)).

Hint: This can be proven by (dyadic) induction; show that $\frac{2n\#}{n\#} = \frac{(2n-1)\#}{n\#} \le \binom{2n-1}{n}$ *(why?), which can then be bounded by* 2^{2n-2}.

Remark: Indeed, the bound used here is quite weak, as Chebyshev's theorem with $c_2 = 1.1$ *gives* $e^{\vartheta(X)} \ll e^{1.1X} = (3.004\ldots)^X$*. However, Erdös' proof is still of interest as it is only based on "elementary" counting arguments.*

(b) The main body of the proof of Bertrand's postulate now proceeds by supposing to the contrary that there are no primes $n < p \le 2n$ for some n, and using the above facts to obtain

$$\frac{2^{2n}}{2n+1} \le \binom{2n}{n} = \prod_{p^r \mid \binom{2n}{n}} p^r \le \prod_{\substack{p \le \sqrt{2n} \\ p^r \mid \binom{2n}{n}}} p^r \cdot \prod_{\sqrt{2n} < p \le \frac{2n}{3}} p$$
$$\le (2n)^{\sqrt{2n}} \cdot 2^{\frac{4n}{3}}$$

Justify each step!

(c) Complete the proof by showing that the above inequality is impossible for sufficiently large n – find an explicit value, and be as precise with your bounds as possible. In order to finish you will need to check that Bertrand's postulate is true for a certain list of small primes. . . .

Hint: One approach is to first find a value of n such that the inequality is false, and then to prove (using calculus) that the left-side continues to grow faster than the right side.

Problems 5–6 build on MV 2.2.1, which uses an alternative approach (based on least common multiples and integrals) to prove the lower bound in (1) with $c_1 = 0.80\cdots$.

5. (a) MV 2.2.1(a). This result was mentioned in class; here $[n_1, \ldots, n_r]$ means the least common multiple of the integers n_j.

(b) Suppose that $\ell := [n_1, \ldots, n_r]$. Prove that

$$\gcd\left(\frac{\ell}{n_1}, \cdots, \frac{\ell}{n_r}\right) = 1.$$

Hint: Suppose that $d \mid \frac{\ell}{n_j}$ *for all* j*, and show that* d *must divide* 1.

(c) Prove Bezout's Identity: If $n_1, \ldots, n_r$ are integers, then there is a linear combination (with some $x_j \in \mathbb{Z}$) such that

$$\gcd(n_1, \ldots, n_r) = x_1 n_1 + \cdots + x_r n_r.$$

Hint: Let g denote the left side, and first show that g divides the right side. Then define a to be the minimum positive value in the set of all linear combinations $\{x_1 n_1 + \cdots + x_r n_r \mid x_j \in \mathbb{Z}\}$. Show that a must divide each n_j, and conclude that a divides g.

(d) MV 2.2.1(b) and (c). The first part is immediate, but for the second you will need to use parts (b) and (c) from above.

6. (a) MV 2.2.1(d). You should find that the maximum value of f occurs at $x = \frac{\phi}{\sqrt{5}} = \frac{\phi}{2\phi - 1}$, where $\phi := \frac{\sqrt{5}+1}{2}$ is the Golden Ratio.

   (b) MV 2.21(e) and (f).

   (c) Conclude that the lower bound in (1) holds with $c_1 = \frac{1}{2}\log 5$.

In lecture we proved Dirichlet's divisor bound (recall $\sigma_k(n) := \sum_{d|n} d^k$):

$$\sum_{n \geq X} \sigma_0(n) = X \log X + (2\gamma - 1)X + O\left(X^{\frac{1}{2}}\right);$$

we typically interpret the main term as saying that on "average" $\sigma_0(n) \sim \log n$. In Problems 7–8 you will prove MV 2.1.12, which states that

$$\sum_{n \geq X} \sigma_1(n) = \frac{\pi^2}{12} X^2 + O(X \log X). \tag{2}$$

As you will see below, this can also be interpreted as giving an "average" value of $\sigma_1(n) \sim \frac{\pi^2 n}{6}$.

7. The first approach is to use the Hyperbola Method as in MV Section 2.1.

   (a) If $y \leq X$ (the specific value will be picked later), explain the identity

$$\sum_{n \leq X} \sigma_1(n) = \sum_{d \leq y} d \left\lfloor \frac{X}{d} \right\rfloor + \sum_{d' \leq \frac{X}{y}} \sum_{d \leq \frac{X}{d'}} d - \sum_{d' \leq \frac{X}{y}} \sum_{d \leq y} d. \tag{3}$$

   (b) Prove that (3) can be evaluated and bounded as follows, where the three parenthetical groupings correspond to the three sums:

$$\left(Xy + O\left(y^2\right)\right) + \left(\frac{\pi^2 X^2}{12} + O(X \log X) + O(Xy)\right) + \left(O(Xy)\right).$$

   (c) Pick an appropriate value of y in order to obtain (2).

8. The second approach uses partial summation and the fact that $\frac{\sigma_1(n)}{n}$ has a convenient form.

(a) Show that $\sum_{n \le X} \frac{\sigma_1(n)}{n} = \sum_{d \le X} \frac{1}{d} \left\lfloor \frac{X}{d} \right\rfloor$, and use this to conclude that

$$\sum_{n \le X} \frac{\sigma_1(n)}{n} = \frac{\pi^2}{6} X + O(\log X).$$

Remark: Dividing both sides by X *shows that* $\frac{\sigma_1(n)}{n} \sim \frac{\pi^2}{6}$ *on "average".*

(b) Now use partial summation to prove (2).

9. In lecture we proved MV Theorem 2.11, which states that

$$\log(\sigma_0(n)) \le \frac{\log n}{\log\log n}\left(\log 2 + O\left(\frac{1}{\log\log n}\right)\right).$$

In this problem you will use the Prime Number Theorem to prove that this bound is **sharp**, i.e., that

$$\limsup_{n \to \infty} \frac{\sigma_0(n)}{\log n / \log\log n} = \log 2.$$

(a) For some large X, let $n_X := \prod_{p \le X} p$. Use PNT to prove that $\log n_X \sim X$.

(b) Now use PNT to show that $\log \sigma_0(n_X) \sim \log 2 \frac{X}{\log X}$.