MATH 7230 Homework 5 - Spring 2017

Due Thursday, Mar. 1 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

Problems

1. MV 1.2.5. Along with a useful general result, this problem gives an interesting (and important!) example of a function that does not have a convergent Dirichlet series.

2. It is a fundamental fact most "reasonable" functions agree with their expansions as Taylor series and Fourier series on some domain (or at least, the class of functions for which this is true is relatively "large", say, in a categorical sense). In contrast, in this problem you will show that the class of functions that can be represented by Dirichlet series is more restricted.

   (a) Suppose that $p(s) := c_n x^n + \cdots + c_0 \in \mathbb{C}[s]$ is a polynomial. Show that there is **no** Dirichlet series $\alpha(s) = \sum\limits_{n\geq 1} \frac{a_n}{n^s}$ such that $p(s) = \alpha(s)$ on some half-plane $\{\mathrm{Re}(s) > \sigma\}$.

   Hint: Recall part (a) from Problem 1.

   (b) We will prove later in the semester that the Riemann zeta function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, so $\zeta(s) = \frac{1}{s-1} + R(s)$, where $R(s)$ is analytic in some neighborhood of $s = 1$. Is this simple pole representable by a Dirichlet series? In other words, is there an $\alpha(s) = \sum\limits_{n\geq 1} \frac{a_n}{n^s}$ such that $\frac{1}{s-1} = \alpha(s)$ on some half-plane?

3. The basic identity that underlies the theory of Fourier series is

$$\frac{1}{2\pi i}\int_0^1 e^{2\pi i \gamma x} = \begin{cases} 1 & \text{if } \gamma = 0; \\ 0 & \text{if } \gamma \in \mathbb{R} \setminus \{0\}. \end{cases}$$

   This is extremely useful because it expresses the *indicator* function $\mathbf{1}_{\gamma=0}$ as a contour integral. For Dirichlet series, the situation is slightly different, as one instead constructs an **approximate** indicator function, and isolating a single coefficient requires a limit. In this problem you will prove that if $\alpha(s) = \sum\limits_{n\geq 1} \frac{a_n}{n^s}$, then

$$\lim_{T\to\infty} \frac{1}{2iT} \int_{\sigma-iT}^{\sigma+iT} \alpha(s) n^s ds = a_n, \tag{1}$$

   where $\sigma > \sigma_a$, the abscissa of absolute convergence for α.

(a) Show that the integral in (1) can be rewritten as

$$in^{\sigma} \sum_{m \geq 1} \frac{a_m}{m^{\sigma}} \int_{-T}^{T} \left(\frac{n}{m}\right)^{it} dt.$$

Note that the sum and integral have been interchanged – you will need to justify this!

(b) Now evaluate the integral in part (a). If you then divide by $2T$, explain why this gives an "approximate" indicator function.

(c) Complete the proof by evaluating the limit.

4. Theorem 2.2 in Montgomery-Vaughan shows that the asymptotic density of the square-free integers is $\frac{6}{\pi^2}$. Write out the details of the proof; you may refer to the textbook, but the goal is for you to understand every step, and in particular, to recognize this as a *sieving* argument.

5. MV 3.2.3. If you completed Problem 4, think about how it is related to this one.