MATH 7230 Homework 6 - Spring 2017

Due Thursday, Mar. 8 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

In the proof of Selberg's upper-bound sieve (MV Theorem 3.2) the main term required minimizing a quadratic expression subject to a linear constraint. Problems 1–3 provide several approaches to solving such optimization questions. In each of these problems the goal will be to find the minimum value of

$$G(\tau) := \sum_{i=1}^{N} a_i \tau_i^2, \qquad a_i \geq 0 \tag{1}$$

such that

$$\sum_{i=1}^{N} b_i \tau_i = 1, \qquad b_i \in \mathbb{R}. \tag{2}$$

The solution is $G(\tau) \geq C := \left(\sum_i \frac{b_i^2}{a_i} \right)^{-1}$, which is achieved when $\tau_i = \frac{b_i}{a_i} C$.

1. The first approach is the most direct, as it attempts to complete the square in (1) while making use of (2).

   (a) For a constant c (to be determined later), write

   $$G(\tau) - c = \sum_{i=1}^{N} \left(a_i \tau_i^2 - c b_i \tau_i \right)$$

   and complete the square in each summand.

   (b) You should have found

   $$G(\tau) = \sum_{i=1}^{N} a_i \left(\tau_i - \frac{c b_i}{2 a_i} \right)^2 + c - \frac{c^2}{4} \sum_{i=1}^{N} \frac{b_i^2}{a_i}.$$

   The minimum value of this expression occurs when all of the quadratic terms are 0, but this is only useful if it is possible to choose τ_i satisfying (2). Assuming that $\tau_i = \frac{c b_i}{2 a_i}$, what can you conclude about c? In particular, how does c relate to C above?

   (c) Now simplify $c - \frac{c^2}{4} \sum_{i=1}^{N} \frac{b_i^2}{a_i}$ to find the minimum value of G.

2. The second approach uses Lagrange Multipliers, which is a general technique for solving constrained optimization problems.

   (a) Define

$$L(\tau, \lambda) := G(\tau) - \lambda\left(\sum_{i=1}^{N} b_i\tau_i - 1\right)$$

   and calculate the partial derivatives

$$\frac{\partial}{\partial \tau_i} L(\tau, \lambda) \qquad \text{and} \qquad \frac{\partial}{\partial \lambda} L(\tau, \lambda).$$

   (b) It is a fundamental fact in the theory of Lagrange Multipliers that the maxima/minima occur at the critical points of $L(\tau, \lambda)$ (to understand this, consider the level curves $G(\tau) = A$; any maxima or minima must occur at values of A such that the level curve is **tangent** to the constraint equation (2)). Find the critical point(s) by solving the system

$$\frac{\partial}{\partial \tau_i} L(\tau, \lambda) = 0,\ 1 \le i \le N, \qquad \text{and} \qquad \frac{\partial}{\partial \lambda} L(\tau, \lambda).$$

   (c) Plug in to determine the value of $G(\tau)$ at any critical points (and compare to Problem 1 if you did it). Why can you conclude that you have found the minimum value of $G(\tau)$?

3. The final approach is to use the Cauchy-Schwarz inequality, which states that for real numbers $\{a_i\}, \{b_i\}$,

$$\sum_{i=1}^{N} a_i b_i \le \left(\sum_{i=1}^{N} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} b_i^2\right)^{\frac{1}{2}},$$

   with equality if and only if the vector $(a_1, \ldots, a_N)$ is a scalar multiple of $(b_1, \ldots, b_N)$.

   (a) Starting with the constraint (2), write

$$1 = \sum_{i=1}^{N} b_i\tau_i = \sum_{i=1}^{N} \frac{b_i}{\sqrt{a_i}} \cdot \sqrt{a_i}\tau_i.$$

   Now apply the Cauchy-Schwarz inequality.

   (b) In order to achieve the lower bound, you will need to use the equality condition. Show that this implies that there is some constant c' such that $c' \frac{b_i}{\sqrt{a_i}} = \sqrt{a_i}\tau_i$ for all i.

   (c) Find an expression for c' in terms of the a_i and b_i, and compare to C.

In Problems 4–5 you will answer a more general version of MV 3.2.5, which is based on Section 2 of Hensley's 1978 paper "An Almost-Prime Sieve". One of the main results that you will prove uses the Basic Sieve Bound for Primes along with corollaries of Selberg's Sieve in order to give upper bounds for the number of integers that are the product of at most 2 primes.

Recall the Basic Sieve Bound for Primes, which states that

$$\pi(X+Y) - \pi(X) \leq \omega(P) + \underbrace{S(X,Y;P)}_{:=\#\{X<n\leq X+Y \mid (n,P)=1\}},$$

where typically $P := \prod_{p\leq z} p$. The statistic $\Omega(n)$ is the **total** number of prime divisors of n (with multiplicity), and $\omega(n)$ is the number of **distinct** prime divisors.

4. Let

$$N_k(X,Y) := \#\{X < n \leq X+Y \mid \Omega(n) \leq k\}.$$

The overall goal is to estimate $N_2(X,Y)$ by sieving with P (this is essentially MV 3.2.5(a)).

(a) Prove that

$$N_2(X,Y) = \#\{X < n \leq X+Y \ : \ \Omega(n) \leq 2, (n,P) = 1\} + \sum_{p|P} \sum_{\substack{X<n\leq X+Y \\ p|n,\ \Omega(n)\leq 2}} \frac{1}{\omega((n,P))}.$$

In particular, explain why the weights are necessary in the second sum to precisely count those n with divisors from P.

(b) Conclude the upper bound

$$N_2(X,Y) \leq S(X,Y;P) + \omega(P) + \sum_{p|P} \left(\pi\left(\frac{X+Y}{p}\right) - \pi\left(\frac{X}{p}\right)\right). \tag{3}$$

Hint: If you make the change of variables $n = pn'$ in the second sum of part (a), then the condition becomes $\Omega(n') \leq 1$. Separate the case 0 and 1....

5. This Problem is essentially MV 3.2.5(b), and is a continuation of Problem 4. Set $P := \prod_{p\leq\sqrt{Y}} p$.

(a) Show that the first two terms of (3) are bounded by

$$\frac{2Y}{\log Y} + O\left(\frac{Y}{(\log Y)^2}\right).$$

(b) Show that the last term of (3) is bounded by

$$\sum_{p\leq\sqrt{Y}} \frac{2\frac{Y}{p}}{\log\left(\frac{Y}{p}\right)} + O\left(\frac{\frac{Y}{p}}{\log\left(\frac{Y}{p}\right)^2}\right). \tag{4}$$

Furthermore, write the first sum above as

$$2Y \sum_{p\leq\sqrt{Y}} \frac{\frac{1}{p}}{\log\left(\frac{Y}{p}\right)} = \frac{2Y}{\log Y} \sum_{p\leq\sqrt{Y}} \frac{1}{p} + \sum_{p\leq\sqrt{Y}} \frac{1}{p}\left(\frac{1}{\log Y - \log p} - \frac{1}{\log Y}\right).$$

Conclude that (4) is

$$\frac{2Y}{\log Y}\sum_{p\le\sqrt{Y}}\frac{1}{p}+O\left(\frac{Y}{(\log Y)^2}\sum_{p\le\sqrt{Y}}\frac{\log p}{p}\right).$$

Remark: If you instead uniformly bound the denominators in (4) *by* $\left(\log\sqrt{Y}\right)^{-1}$ *(since* $p\le\sqrt{Y}$*), you obtain an overall constant of* 2 *instead of* 4.

(c) Finally, conclude that

$$N_2(X,Y)\le\frac{2Y\log\log Y}{\log Y}+O\left(\frac{Y}{\log Y}\right).$$

Remark: To get the constant of 2*, you will need MV Theorem 2.7(b) and (d), which rely on Chebyshev's Theorem. Note that up until this point every bound was derived solely from Sieve methods!*