MATH 7230 Homework 7 - Spring 2017

Due Thursday, Mar. 22 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

Problems 1–4 fill in the details of the proof of the main term in MV Theorem 3.10, where Selberg's (upper-bound) sieve is used to bound the number of Twin Primes.

1. Recall the *divisor function* $\sigma_0(n) := \sum_{d|n} 1$. The main term of Selberg's sieve for Twin Primes requires the asymptotic formula

   $$\sum_{n\leq Z} \frac{\sigma_0(n)}{n} = \frac{(\log Z)^2}{2} + O\left(\log Z\right).$$

   Prove this.

   Hint:Theorem 2.3 showed that $\sum_{n\leq Z} \sigma_0(n) = Z\log Z + O(Z)$*. Now use partial summation. . . .*

2. Selberg's sieve (for arbitrary collections of restricted residue classes) requires minimizing a quadratic expression subject to a linear constraint. This is typically simplified by diagonalizing the quadratic form through a linear change of variables of the form

   $$y_f = \sum_{f|d|P} \frac{\Lambda_d b(d)}{d}.$$

   It is important that this is an **invertible** linear map. In this problem you will verify this by showing a *(doubly) bounded Möbius inversion* formula.

   In particular, suppose that $\{a_n\}, \{b_n\}$ are sequences such that $a_f = \sum_{f|d|P} b_d$. Prove that

   $$b_d = \sum_{d|f|P} a_f \cdot \mu\left(\frac{f}{d}\right).$$

   Hint: One approach is to directly plug in the formula for a_f *and simplify, using properties of the Möbius* μ*-function. Can you find a proof that uses Dirichlet series and/or convolution?*

3. In lecture I skipped the derivation of the Twin Prime Constant in MV Theorem 3.10; the next two problems fill in those missing details (see HW 2 #5 for a heuristic derivation, and note that an earlier typo – a missing factor of 2 – has been corrected). The main

term requires the asymptotic behavior of $L = \sum_{f|P,\, f \le z} \mu(f)^2 g(f)$, where (for squarefree f)

$$g(f) = \prod_{p|f} \frac{b(p)}{p - b(p)}, \qquad \text{and} \qquad b(p) = \begin{cases} 1 & \text{if } p = 2, \\ 2 & \text{otherwise.} \end{cases}$$

It is difficult to work with g directly, and one sees that it should be a relatively good approximation to instead take

$$g(f) \to \prod_{p|f} \frac{b(p)}{p} = \frac{2^{\omega(f)-\delta}}{f} = \frac{\sigma_0(f)}{2^\delta f},$$

where δ is the indicator function for $2 \mid f$. More precisely, it is particularly convenient to introduce a convolved function when approximating g. Define $c(m)$ by the relation

$$\mu(m)^2 g(m) = \left(\frac{\sigma_0}{\mathrm{id}} \star c\right)(m) = \sum_{d|m} \frac{\sigma_0(d)}{d} c\left(\frac{m}{d}\right). \tag{1}$$

(a) Using basic properties of Dirichlet series, explain why

$$\alpha_c(s) := \sum_{k \ge 1} \frac{c(k)}{k^s} = \sum_{m \ge 1} \frac{\mu(m)^2 g(m)}{m^s} \left(\sum_{n \ge 1} \frac{\sigma_0(n)}{n^{s+1}} \right)^{-1}.$$

(b) Now use the multiplicative properties of the functions on the right to conclude that

$$\alpha_c(s) = \left(1 + \frac{1}{2^s}\right) \prod_{p>2} \left(1 + \frac{2}{(p-2)p^s}\right) \prod_p \left(1 - \frac{1}{p^{s+1}}\right)^2. \tag{2}$$

Hint: For the second product, recall equation (1.5) in MV, which uses the fact that $\sigma_0 = 1 \star 1$ to express the Dirichlet series for σ_0 in terms of $\zeta(s)$.

(c) Multiply and group terms to show that for $p > 2$ the expression in the product is of the form

$$1 + O\left(\frac{1}{p^{s+2}}\right) + O\left(\frac{1}{p^{2s+2}}\right) + O\left(\frac{1}{p^{3s+3}}\right), \tag{3}$$

where the constants all have a uniform bound.

Conclude that $\alpha_c(s)$ converges absolutely for $\mathrm{Re}(s) > -\frac{1}{2}$. Which part of (3) imposes this restriction? Also, be sure to show that the $p = 2$ terms in (2) also converge in this half-plane.

4. This problem finishes the final bounds for the main term; if you did not complete Problems 1–3, you may still use the results.

(a) Using (1), show that

$$L = \sum_{k \le Z} c(k) \sum_{n \le \frac{Z}{k}} \frac{\sigma_0(n)}{n}.$$

(b) Using Problem 1, show that

$$L = \frac{1}{2}(\log Z)^2 \sum_{k\le Z} c(k) + O\left(\log Z \sum_{k\le Z} |c(k)| \log k\right) + O\left(\sum_{k\le Z} |c(k)| (\log k)^2\right). \quad (4)$$

(c) Using Problem 3 (c), show that for any $\delta > 0$, the two big-O terms in (4) are $O(\log Z)$ and $O(1)$, respectively.

Hint: Recall that $\log k \ll k^{\varepsilon}$ for any $\varepsilon > 0$, and compare the sums to $\alpha_c(-\varepsilon)$.

(d) Finally, show that for any $\delta > 0$,

$$\sum_{k\le Z} c(k) = \alpha_c(0) + O\left(\frac{1}{Z^{\frac{1}{2}-\delta}}\right),$$

and

$$\alpha_c(0) = \left(2 \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\right)^{-1}, \quad (5)$$

where the product is the *Twin Prime Constant.*

Hint: For the first part, use Problem 3 (c). In particular, write the missing tail sum as

$$\sum_{k\ge Z} c(k) = \sum_{k\ge Z} c(k) \frac{k^{\frac{1}{2}-\delta}}{k^{\frac{1}{2}-\delta}} \le \frac{1}{Z^{\frac{1}{2}-\delta}} \sum_{k\ge Z} \frac{c(k)}{Z^{-(\frac{1}{2}-\delta)}}.$$

5. In this problem you will answer MV 3.4.2, which uses Selberg's sieve to provide an "upper bound to Goldbach's Conjecture". If you solved Problem 3, this should be fairly straightforward. You may assume/use any bounds from the proof of MV Theorem 3.10.

For a (large) even integer $2n$, we call (p_1, p_2) a *Goldbach pair for* $2n$ if p_j is prime and $p_1 + p_2 = 2n$.

(a) As usual, let $P := \prod_{p\le Z} p$. Explain why $m \in [Z+1, 2n-Z-1]$ can **not** be part of a Goldbach pair if $(m, P) > 1$ or $(2n-m, P) > 1$.

(b) Show that part (a) allows for use of Selberg's upper-bound sieve with excluded residue classes $m \not\equiv 0, 2n \pmod p$. Conclude that the basic formulas of Selberg's sieve apply with

$$b(p) = \begin{cases} 1 & \text{if } p \mid 2n, \\ 2 & \text{otherwise.} \end{cases}$$

(c) Finally, prove that the number of Goldbach pairs for $2n$ is at most

$$8c' \frac{2n}{(\log 2n)^2}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right), \quad (6)$$

where

$$
\begin{aligned}
c' &:= \prod_{p|2n} \frac{p-1}{p} \prod_{p\nmid 2n} \frac{p-2}{p} \prod_{p} \left(1 - \frac{1}{p}\right)^{-2} \\
&= \prod_{p|2n} \left(1 + \frac{1}{p-1}\right) \prod_{p\nmid 2n} \left(1 - \frac{1}{(p-1)^2}\right).
\end{aligned}
$$

(d) There are two additional technical points: First, note that the sieve was defined in terms of primes up to Z (which is chosen to be $\frac{\sqrt{2n}}{\sqrt{\log 2n}}$ in the end), but the final expression in part (c) only uses the prime factorization of $2n$. Does this matter? What happens if $2n$ has a prime factor larger than Z?

Hint: Show that such a prime is absorbed into the error term of (6).

Second, what about the excluded ranges in part (a), $[1, Z]$ and $[2n - Z, 2n]$?