# MATH 7230 Homework 9 - Spring 2017

Due Thursday, Apr. 19 at 10:30

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "MV A.B.C" means Exercise C at the end of Section A.B in the textbook (Montgomery-Vaughan).

All of the problems this week are based on A. Granville's survey "Primes in intervals of bounded lengths", which is linked from the course website.

1. The basic counting function for sieving an admissible $k$-tuple $A = \{a_1, \ldots, a_k\}$ is

$$\omega_A(D) = \omega(D) := \#\{m \bmod D \mid P_A(m) \equiv 0 \pmod{D}\},$$

   where $P_A(n) := (n + a_1)\cdots(n + a_k)$ and $D$ is squarefree. The Chinese Remainder Theorem easily implies that $\omega$ is multiplicative (using the fact that $D$ is squarefree!), so it is only necessary to determine $\omega(p)$.

   (a) Pick an admissible 4 or 5-tuple and calculate $\omega(p)$ for all $p$.

   (b) Prove that in general, $\omega(p) = k$ for all but finitely many $p$.

2. Verify the "Reciprocity Law" from Section 4.5 of Granville's survey. In particular, suppose that $\{L(d)\}_{d=1}^{\infty}$ a sequence defined on squarefree natural numbers, and define

$$Y(r) := \mu(r) \sum_{\substack{m \le R \\ m \text{ squarefree} \\ r \mid m}} L(m).$$

   Prove that

$$L(d) = \mu(d) \sum_{\substack{n \le R \\ n \text{ squarefree} \\ d \mid n}} Y(n).$$

3. Granville then applies Problem 2 with $L(d) := \frac{\lambda(d)\omega(d)}{d}$, and obtains (as the main part of the "negative" term in the Goldston-Pintz-Yildirim sieve)

$$S_1 = \sum_{\substack{r,s \le R \\ \text{squarefree}}} Y(r)Y(s) \sum_{\substack{d_1,d_2 \le R \\ \text{squarefree} \\ d_1 \mid r, d_2 \mid s}} \mu(d_1)\mu(d_2)\frac{(d_1, d_2)}{\omega((d_1, d_2))}. \tag{1}$$

   He then states that the interior sum is multiplicative, so it is sufficient to evaluate at primes... but this takes a bit work to see.

In this problem and the next you will simplify a general Möbius sum that is "intertwined" by greatest common divisors. In particular, let

$$F_{r,s} := \sum_{\substack{d_1|r \\ d_2|s}} \mu(d_1)\mu(d_2) f((d_1, d_2)),$$

where $f$ is an arbitrary function. Note that the $\mu$-function guarantees throughout that $d_1$ and $d_2$ are squarefree, and thus you may as well assume that $r$ and $s$ are also squarefree (if not, simply replace $r$ by its *squarefree part*, $\mathrm{sqfree}(r) := \prod_{p|r} p$).

(a) Let $G := (r, s)$, and write $r = Gr', s = Gs'$ with $(r', s') = 1$. Furthermore, if $g \mid G$, let $h := \frac{G}{g}$. Show that

$$F_{r,s} = \sum_{g|G} f(g) \sum_{\substack{d_1'|hr',\ d_2'|hs' \\ (d_1', d_2')=1}} \mu(d_1')\mu(d_2').$$

(b) Show that the inner sums can be rewritten as

$$\sum_{\substack{d_1'|hr',\ d_2'|hs' \\ (d_1', d_2')=1}} \mu(d_1')\mu(d_2') = \sum_{d_1'|hr'} \mu(d_1') \sum_{d_2'|\frac{h}{(d_1', h)}s'} \mu(d_2').$$

In particular, why can you conclude that $d_1'$ has no common factors with $s'$?

(c) The basic summation identity for $\mu$ implies that the inner sum vanishes unless $h = (d_1', h)$ and $s' = 1$. Explain why this means that $h \mid d_1'$ and $s \mid r$.

(d) Finally, plug in these conditions to the remaining sums to conclude that

$$F_{r,s} = \begin{cases} \sum_{g|r} f(g)\mu\left(\frac{r}{g}\right). & \text{if } r = s; \\ 0 & \text{if } r \neq s. \end{cases} \tag{2}$$

4. In this problem you will use (2) to simplify (1).

(a) First, plug in directly and show that

$$S_1 = \sum_{r \le R,\ \text{squarefree}} Y(r)^2 \mu(r) \sum_{g|r} \mu(g) \frac{g}{\omega(g)}.$$

(b) Now it is clear that the inner sum is multiplicative. Conclude that

$$\sum_{g|r} \mu(g) \frac{g}{\omega(g)} = \prod_{p|r} \left(1 + \mu(p) \frac{p}{\omega(p)}\right).$$

Simplify to obtain

$$S_1 = \sum_{r \le R,\ \text{squarefree}} Y(r)^2 \frac{\phi_\omega(r)}{\omega(r)},$$

where $\phi_\omega$ is the multiplicative function defined by $\phi_\omega(p) := p - \omega(p)$.

*Remark: The notation is meant to be reminiscient of Euler's totient function $\varphi(p) = p-1$; note that in general $p - \omega(p) \le p - 1$.*

5. Recall the definition of $\omega_A(D)$ from Problem 1, and further define (for any $1 \le j \le k$)

$$\omega_A^*(D) = \omega^*(D) := \#\left\{m \in \Omega(D) \;\mid\; (m + a_j, D) = 1\right\}.$$

(a) Prove that $\omega * (p) = \omega(p) - 1$.

*Hint: Start by observing that $\omega(p) = |A \bmod p\,|$.*

(b) The sieving function introduced by GPY has the form

$$S(A;X) = X\left[\sum_{\substack{d_1,d_2\le R\\ \text{squarefree}\\ D=[d_1,d_2]}} \lambda(d_1)\lambda(d_2)\left(k\frac{\omega^*(D)}{\varphi(D)} - \log 3X\frac{\omega(D)}{D}\right)\right].$$

Show that for any $D$,

$$\frac{\omega^*(D)}{\varphi(D)} \le \frac{\omega(D)}{D}.$$

*Remark: This means that the term in the inner parentheses can never be positive if $\log 3X > k$.*