LSU Problem Solving Seminar - Fall 2018
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Useful facts:

- **Arithmetic-Geometric Mean Inequality.** If $a_1, \dots, a_n$ are non-negative real numbers, then
$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$
Furthermore, the right side is strictly larger than the left unless all of the a_i are equal.

- **Hölder's p-norm Inequality.** If $0 < p < q$ and $a_1, \dots, a_n$ are non-negative real numbers, then
$$\left(\frac{a_1^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{1}{q}},$$
with strict inequality unless all of the a_i are equal.

- **Cauchy-Schwarz Inequality.** If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers, then
$$\left(a_1 b_1 + \cdots + a_n b_n\right)^2 \leq \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right).$$
Furthermore, the right side is strictly larger unless $(b_1, \dots, b_n) = (\lambda a_1, \dots, \lambda a_n)$ for some real λ. Written in vector notation and Euclidean distance, $|\overrightarrow{a} \cdot \overrightarrow{b}|^2 \leq |\overrightarrow{a}|^2 \cdot |\overrightarrow{b}|^2$.

- **Triangle Inequality.** If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers, then
$$\sqrt{\left(a_1 + b_1\right)^2 + \cdots + \left(a_n + b_n\right)^2} \leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}.$$
Written in vector notation, $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

- **Rearrangement Inequality.** Suppose that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. If $a_1', a_2', \dots, a_n'$ is any reordering of $a_1, \dots, a_n$, then
$$a_1' b_1 + a_2' b_2 + \cdots + a_n' b_n < a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Warm Up

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1. For each of the following pairs, determine which expression is larger (without using a calculator!):

   (a) $10^{\sqrt{2}}$ or $\sqrt{2}^{10}$?

   (b) $\left(1 + \left(2 + \left(3 + \left(4 + 5^2\right)^2\right)^2\right)^2\right)^2$ or $\left(5 + \left(4 + \left(3 + \left(2 + 1^2\right)^2\right)^2\right)^2\right)^2$?

   (c) $\sqrt[2018]{2018}$ or $\sqrt[2017]{2017}$?

2. A farmer wishes to construct a rectangular pen using 200 feet of fence. What is the maximum area that he can enclose? Try to solve this **without** using calculus!

   Hint: If the pen has width w and length ℓ, the area is ℓw, with the constraint $2w + 2\ell = 200$. Now apply the Arithmetic-Geometric Mean inequality....

3. Suppose that $t > 0$. Prove that $f(t) = t + t^{-1}$ has a unique minimum value m, i.e., that $f(t_0) = m$ for a unique t_0, and $f(t) > m$ for all other $t \neq t_0$. Determine t_0 and m.

Main Problems

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4. Suppose that n is a positive integer. Consider the following quantities:

$$\begin{aligned} &\Big(1 \cdot 2 + 3 \cdot 4 + \cdots + (2n-1) \cdot 2n\Big)^2, \\ \text{and} \quad &\Big(1^2 + 3^2 + \cdots + (2n-1)^2\Big) \cdot \Big(2^2 + 4^2 + \cdots + (2n)^2\Big). \end{aligned}$$

   (a) Calculate both expressions for $n = 1, 2, 3$.

   (b) Use the Cauchy-Schwarz inequality to determine which is larger in general.

5. [VTRMC **1982 # 4**] Prove that $t^{n-1} + t^{1-n} < t^n + t^{-n}$ when $t \neq 1$, $t > 0$, and n is a positive integer.

   Hint: How does $t^n + t^{-n}$ compare to $\left(t + t^{-1}\right)\left(t^{n-1} + t^{1-n}\right)$? Recall Problem 3.

6. In this problem you will practice using the method of *Lagrangian multipliers* to solve constrained maxima/minima problems. Suppose that $f(x_1, \cdots, x_n)$ is a function, and $g(x_1, \cdots, x_n) = 0$ is a constraint. Define

$$L(x_1, \cdots, x_n, \lambda) := f(x_1, \cdots, x_n) - \lambda \cdot g(x_1, \cdots, x_n).$$

   Then the maxima/minima of f subject to the constraint $g = 0$ occur at the critical points of L; these are the points where all partial derivatives vanish,

$$\frac{\partial L}{\partial x_1}, \ \cdots, \ \frac{\partial L}{\partial x_n}, \ \frac{\partial L}{\partial \lambda} = 0.$$

   (a) As a first example, solve Problem 2 above using Lagrangian multipliers. Suppose that the pen has width w and length ℓ. Then the area is $f(w, \ell) = w\ell$, and the constraint is $g(w, \ell) = 2w + 2\ell - 200 = 0$ (why?). Now find the critical point(s) of

$$L(w, \ell, \lambda) = w\ell - \lambda\left(2w + 2\ell - 200\right).$$

   (b) Find the maximum and minimum value of $x \cdot y$ among all points on the unit circle around the origin, $x^2 + y^2 = 1$.

7. [VTRMC **2013 # 5**] Prove that $\dfrac{x}{\sqrt{1+x^2}} + \dfrac{y}{\sqrt{1+y^2}} + \dfrac{z}{\sqrt{1+z^2}} \leq \dfrac{3\sqrt{3}}{2}$ for any positive real numbers x, y, z such that $x + y + z = xyz$.

8. (a) [**Gelca-Andreescu 453**] Prove that for all positive integers n,

$$\sqrt[n]{3} + \sqrt[n]{7} > \sqrt[n]{4} + \sqrt[n]{5}.$$

   Hint: How do both sides compare to $2 \cdot \sqrt[n]{4.5}$?

(b) If $0 < x < 1$, determine which is larger:

$$2^x + 7^x \quad \text{or} \quad 4^x + 5^x?$$

Hint: Define $H(y) = y^x + (9-y)^x$.

9. The *floor* of a real number x is defined to be the greatest integer $n \leq x$, and is denoted by $\lfloor x \rfloor$. In other words, $\lfloor x \rfloor = m$, where m is the integer such that

$$m \leq x < m + 1.$$

   (a) Find an example where $2\lfloor x \rfloor \neq \lfloor 2x \rfloor$.

   (b) Prove that for any real number x

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor.$$

   (c) Suppose that n is a positive integer. Prove that $\left\lfloor \sqrt{4n^2 + 4n} \right\rfloor = 2n$.

10. [Putnam **1948 B3**] Show that $\lfloor \sqrt{n} \rfloor + \lfloor \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$ for a positive integer n.