LSU Problem Solving Seminar - Fall 2018
Oct. 31: Post-Exam Review of Virginia Tech Math Contest

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

This week's practice sheet provides a detailed look at several of the problems from last weekend's 2018 Virginia Tech Regional Math Contest. Each contest problem is **preceded** by a related problem that illustrates some relevant techniques in a simpler context.

1. In the first portion of this problem you will evaluate

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx.$$

It is a fact that $I = q\pi\ln(2)$ for some rational number q. The goal of the approach outlined here is to make a substitution that relates I to $-I$ using the fact that $\ln(a^{-1}) = -\ln(a)$.

(a) In particular, show that making the substitution such that u and x are related by $(1+x) = \frac{2}{1+u}$ gives

$$I = \int_1^0 \frac{\ln\left(\frac{2}{1+u}\right)}{\frac{2(1+u^2)}{(1+u)^2}} \frac{(-2)du}{(1+u)^2} = \int_0^1 \frac{\ln(2) - \ln(1+u)}{1+u^2}\,du = \int_0^1 \frac{\ln(2)}{1+u^2}\,du - I.$$

Verify all of the details! Now evaluate the integral in order to find the value of I.

(b) The angle summation formula for the arctangent is typically stated as

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right) \qquad \text{for } |xy| < 1.$$

Suppose that $xy > 1$. Explain why this should be replaced by $\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right) + \pi$.

2. [VTRMC **2018 # 1**] It is known that $\displaystyle\int_1^2 \frac{\arctan(1+x)}{x}\,dx = q\pi\ln(2)$ for some rational number q. Determine q. Here $0 \le \arctan(x) < \frac{\pi}{2}$ for $0 \le x < \infty$.

Hint: First integrate by parts. Then you should be able to apply ideas from Problem 1.

3. (a) Find a real function $f(x)$ such that $f(f(x)) = x$. There is more than one solution! Try to find a few of them on your own before proceeding further in the problem.

Remark: Such a function is called an involution: f is its own inverse.

(b) Show that $f(x) := -\frac{x}{x+1}$ is a function whose domain (and range) consists of all real $x \neq -1$, and $f(f(x)) = x$.

(c) Find all linear functions $f(x) = ax + b$ such that $f(f(x)) = x$.

(d) Show that if $f(x)$ is a linear function and $f(f(x)) = x + a$ for some real number a, then $f(x) = x + \frac{a}{2}$.

4. [VTRMC **2018 # 3**] Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n + 1$. Here $\mathbb{N}$ is the positive integers $\{1, 2, 3, \ldots\}$.

   Hint: You cannot immediately assume that f is a linear function. Instead, start by using the given relation for $n = 1, 2, 3, \ldots$.

5. Suppose that p is prime.

   (a) Show that $\binom{p}{k}$ is a multiple of p for all $1 \le k \le p - 1$.

   (b) If $p \ne 2$, determine the m such that $\binom{2p}{m}$ is divisible by p.

   (c) Show that $\binom{p^2}{k}$ is a multiple of p for all $1 \le k \le p^2 - 1$.

   (d) The *floor* of a real number x is the greatest integer $a \le x$, and is denoted by $\lfloor x \rfloor = a$. Show that the highest power of p that divides $n!$ is

$$r = \sum_{k \ge 1} \left\lfloor \frac{n}{p^k} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots .$$

   In other words, $n!$ is a multiple of p^r, but not of p^{r+1}.

6. [VTRMC **2018 # 4**] Let m, n be integers such that $n \ge m \ge 1$. Prove that $\dfrac{\gcd(m,n)}{n}\dbinom{n}{m}$ is an integer. Here gcd denotes the greatest common divisor, and $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ denotes the binomial coefficient.

7. (a) Suppose that $a > 0$. Determine the value of $\displaystyle\lim_{t \to 0} \frac{1 - \cos(at)}{1 - \cos(t)}$.

   (b) Prove that if $0 < x < 1$, then $1 - \cos(x) \le \dfrac{x^2}{2}$.
   Hint: Recall the Taylor series for $\cos(x)$, and use the estimation for alternating series.

   (c) Use the (finite) geometric summation and Euler's identity $e^{ix} = \cos(x) + i\sin(x)$ to show that

$$\left|1 + e^{it} + \cdots + e^{(n-1)it}\right| = \left|\frac{1 - e^{nit}}{1 - e^{it}}\right| = \left|\frac{1 - \cos(nt)}{1 - \cos(t)}\right|^{\frac{1}{2}} .$$

8. [VTRMC **2018 # 5**] For $n \in \mathbb{N}$, let $a_n = \displaystyle\int_0^{1/\sqrt{n}} \left|1 + e^{it} + e^{2it} + \cdots + e^{nit}\right| dt$. Determine whether the sequence $(a_n) = a_1, a_2, \ldots$ is bounded.

9. The n-th *Harmonic number* is $H_n := 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$.

   (a) Show that $H_n > \displaystyle\int_1^n \frac{1}{x}\, dx = \ln(n)$.

(b) Similarly, show that $H_n < \ln(n) + 1$.

(c) Parts (a) and (b) show that the main asymptotic term for H_n is given by the natural logarithm, i.e., that $H_n \sim \ln(n)$ for large n. In fact, if you are much more careful with the integral comparisons it is possible to show that

$$H_n = \ln(n) + \gamma + R_n,$$

where $\gamma \sim 0.577\ldots$ is the *Euler-Mascheroni constant*, and $|R_n| < \frac{1}{n}$.

(d) Show that $1 + \dfrac{1}{3} + \dfrac{1}{5} + \cdots + \dfrac{1}{2n-1} \sim \dfrac{\ln(n)}{2} + \ln(2)$.
Hint: Write the sum in terms of H_{2n} and H_n.

10. [VTRMC **2018 # 6**] For $n \in \mathbb{N}$, define $a_n = \dfrac{1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}}{n+1}$ and $b_n = \dfrac{\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}}{n}$. Find the maximum and minimum of $a_n - b_n$ for $1 \leq n \leq 999$.
Hint: Write a_n and b_n in terms of Harmonic numbers.

11. Let $f(x)$ be the "triangle function" given by

$$f(x) := \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}; \\ 2(1-x) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Give a precise description of $f(f(x))$, $f(f(f(x)))$, and $f^n(x)$, which is f iterated n times. How many peaks does the graph of $f^n(x)$ have?

12. [VTRMC **2018 # 7**] A continuous function $f : [a,b] \to [a,b]$ is called piecewise monotone if $[a,b]$ can be subdivided into finitely many subintervals

$$I_1 = [c_0, c_1], I_2 = [c_1, c_2], \ldots, I_\ell = [c_{\ell-1}, c_\ell]$$

such that f restricted to each interval I_j is strictly monotone, either increasing or decreasing. Here $c_0 = a$ and $c_\ell = b$, and each I_j is a maximal interval on which f is strictly monotone. Such a maximal interval is called a *lap* of the function, and $\ell = \ell(f)$ is the *lap number* of f. Show that the sequence $\left(\sqrt[n]{\ell(f^n(x))}\right)$ converges.