# LSU Problem Solving Seminar - Fall 2018
## Nov. 7: Probability

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Useful facts:

- **Probability Spaces.** A (countable) probability space consists of a set of distinct events $A_1, A_2, \ldots$ and a probability function $0 \leq p \leq 1$ such that $p(A_1) + p(A_2) + \cdots = 1$.

  In a finite probability space, typically $p(A) = \dfrac{\#\text{ outcomes in } A}{\#\text{ total outcomes}}$. For example, if two dice are rolled, there are 5 ways to obtain a sum of 6 (namely, $5+1, 4+2, 3+3, 2+4, 1+5$), and $6^2 = 36$ total combinations, so $p(6) = \frac{5}{36}$.

- **Random Variables and Expectation.** A random variable $X$ assigns a real value $x$ to each event $A$. The expected value, or *average* of $X$ is
$$E[X] := \sum_x x \cdot P(X = x).$$
  For example, the expected number of Tails when two coins are flipped is $0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4} = 1$.

- **Additivity of Expectation.** If $X$ and $Y$ are random variables, $E[X+Y] = E[X] + E[Y]$.

- **Exponential and Stirling approximation.** Use the following formulas to approximate discrete probabilities for large $n$ (and small $k$):
$$\left(1 - \tfrac{1}{n}\right)^n \sim e^{-1}, \qquad n! \sim \left(\tfrac{n}{e}\right)^n, \qquad \text{and} \qquad \tbinom{n}{k} \sim \tfrac{n^k}{k!}.$$

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Warm Up

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1. At LSU, 83% of students own an article of Purple or Gold clothing, 65% have an LSU Notebook, and 58% have a Mike the Tiger Bobblehead*. Prove that there is at least one student who has items in all three categories.

   * Statistics may not be accurate.

2. (a) Suppose that two coins are flipped. What is the expected (i.e. "average") number of Heads?

   (b) What is the expected number of Heads when 5 coins are flipped?

3. Suppose that three dice are rolled.

   (a) What is the probability that the sum is 12?

   (b) What is the probability that the sum is even?

   (c) What is the probability that the sum is 11 or greater?

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Main Problems

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4. What is the expected number of Heads when $n$ coins are flipped?

   *Remark: A direct solution requires finding a formula for* $\sum_{k=0}^{n} k \binom{n}{k} x^k$. *An easier probabilistic solution uses properties of expectation. . . .*

5. In a standard pair of dice, the faces of each cube are labeled $1, 2, 3, 4, 5, 6$. In a pair of *Sicherman* dice, the first dice is labeled by $1, 2, 2, 3, 3, 4$ and the second dice by $1, 3, 4, 5, 6, 8$.

   (a) What is the probability of rolling a total of 6 with standard dice? What is the probability of rolling 6 with Sicherman dice?

   (b) Prove that Sicherman dice has the same distribution as standard dice: in other words, the probability of rolling **any** total is the same with either pair of dice!

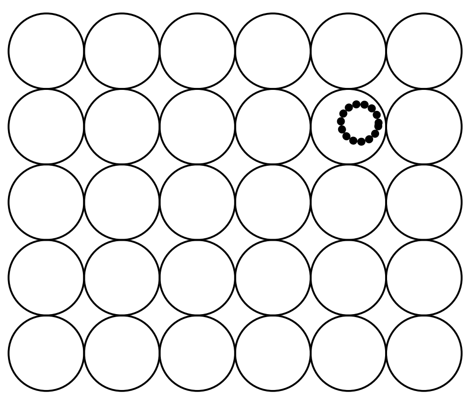
   *Hint: How does the distribution of standard dice relate to the polynomial product*

   $$\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2?$$

   *In fact, it can be shown that the Sicherman dice are the* **only** *alternative pair of six-sided dice with positive labels that are equivalent to standard dice.*

6. A common style of Carnival Game is a Ball-in-Cup Throwing Game. A large number of cups with a 4-inch diameter are placed in a square array, as pictured. The Ball is a sphere with a 2-inch diameter, and the game is won if it successfully lands in a cup.

   (a) Calculate the probability of winning the game. Make (and justify!) the reasonable physical assumptions that the Ball falls essentially vertically in a random position in the array of Cups, and that it lands in a Cup if and only if widest portion of the Ball lies entirely within the Cup – see figure.



   (b) More realistically, the dimensions might be Cups with a 4-inch diameter and a Ball with a 3-inch diameter. What is the probability of winning in this case? If it costs \$2 to play, and the prize is a novelty stuffed animal that is worth at most \$20, should you play the game?

7. A picnic table seats 5 people along each side. At a cookout, 5 men and 2 women sit at random positions around the table. In this problem you will determine the probability that the women are on opposite sides.

(a) First, count all of the possibilities directly. This is simplified somewhat by assuming that all of the men are indistinguishable, and likewise for the women. Show that there are $\binom{10}{5}\binom{5}{2}$ total configurations. How many of them have the women on opposite sides?

(b) Now use "Sample Space Reduction" – suppose that the women sit down first. Then the calculation is very simple, and the men can be ignored!

8. A standard deck of 52 cards is shuffled into a random order. Note that there are 52! possible orderings.

   (a) What is the probability that the Ace of Spades is the first card?

   (b) What is the probability that the Ace of Spades is the 21-st card?

   (c) What is the probability that the first 13 cards are all Spades?

   (d) What is the probability that the first half of the deck contains the same number of Black cards as the second half?

9. [Putnam **2007 A3**] Let $k$ be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.