Important upcoming dates:

- The Problem-Solving Seminar will **not** meet on Wednesday, Nov. 21 due to the Thanksgiving holiday. The last meeting of the semester will be Wednesday, Nov. 28.
- Putnam Mathematical Competition: **Sat., Dec. 1.** The Exam will take place in Lockett 232 from 8:30 A.M. – 5:00 P.M.

LSU Problem Solving Seminar - Fall 2018

Nov. 14: Polynomials and Complex Numbers

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Let $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ be a polynomial with real coefficients. It is *monic* if the leading coefficient $a_n = 1$. The *degree* of a polynomial is the exponent of the leading term, in this case n. A *root* of f is a value r such that $f(r) = 0$.

- **Rational Roots Test.** If all of the a_i are integers and $r = \frac{p}{q}$ is a root, then p is a divisor of a_0 and q is a divisor of a_n.
- **Descartes' Rule of Signs.** If the non-zero coefficients of $f(x)$ change sign s times, then f has at most s positive roots (with multiplicity). The actual number of positive roots is less than s by some multiple of 2. Replacing x by $-x$ gives a similar test for negative roots.
- **Polynomial Division Algorithm.** A polynomial $f(x)$ is a *multiple* of $g(x)$ if $f(x) = h(x) \cdot g(x)$ for some polynomial $h(x)$. If $f(x)$ is not a multiple of $g(x)$, then there are polynomials $q(x)$ ("quotient") and $r(x)$ ("remainder") such that $f(x) = q(x) \cdot g(x) + r(x)$, where $r(x)$ has lower degree than $g(x)$.
- **Repeated Roots.** A polynomial $f(x)$ is divisible by $(x-r)^k$ (i.e. the root r has *multiplicity* at least k) if and only if $f(r) = 0, f'(r) = 0, \ldots, f^{(k-1)}(r) = 0$.
- **Fundamental Theorem of Algebra.** A polynomial of degree n has exactly n complex roots, counted with multiplicity. In particular, it has at most n real roots. Furthermore, if the roots are $r_1, \ldots, r_n$, then $f(x) = c(x - r_1) \cdots (x - r_n)$ for some constant c.
- **Sum and Product of Roots.** If a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has roots (with repetition) $r_1, \ldots, r_n$, then

$$a_{n-1} = -(r_1 + \cdots + r_n); \qquad a_0 = (-1)^n r_1 \cdots r_n.$$

- **Roots of Unity.** The roots of $x^n - 1$ are $1, e^{\frac{2\pi i}{n}}, e^{\frac{2\cdot 2\pi i}{n}}, \ldots, e^{\frac{(n-1)\cdot 2\pi i}{n}}$. These can also be written as $1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}$, where $\zeta_n := e^{\frac{2\pi i}{n}}$. The previous property implies that

$$1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1} = 0.$$

- **Euler's Formula.** For real x, $e^{ix} = \cos(x) + i\sin(x)$.

Warm Up

1. Find all real roots (with multiplicity) of the following polynomials:
   (a) $x^3 - 3x + 2$,
   (b) $2x^4 + x^3 - 7x^2 - 3x + 3$,
   (c) $5x^6 + 13x^4 + 8x^2 + 29$.

2. Two of the roots of the polynomial $p(x) = x^4 - 9x^2 + 4x + 12$ are $x = 2$ and $x = -3$. **Without** performing any polynomial division, determine the other roots and complete factorization of $p(x)$.

3. (a) Determine the number of real roots of the cubic polynomial $2x^3 + x - 1$.
   (b) Are there any real values of a such that $ax^3 + x - 1$ has three real roots?

Main Problems

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4. (a) One of the following polynomials is a multiple of $x^3 - x$, and the other is a multiple of $x^3 - 1$; determine which is which:

$$x^{2018} - x^{218} + x^{28} - x^2, \qquad x^{2018} - x^{208} + x^{21} - x^8 + x - 1.$$

   (b) Determine the remainders when the opposite polynomial divisions are performed. For example, if $f(x)$ is not a multiple of $x^3 - x$, then

$$f(x) = \left(x^3 - x\right) q(x) + r(x),$$

   where $q(x)$ is a polynomial, and $r(x)$ is a non-zero polynomial of degree at most 2. (If $r(x) = 0$, this simply means that f was a multiple of $x^3 - x$).

5. There are many settings in mathematics where one begins with a polynomial $f(x)$ and attempts to determine its roots. However, in this problem, you will consider this question in reverse: given a real number α, is there a (simple) polynomial such that $f(\alpha) = 0$?

   (a) Consider the two real numbers

$$\alpha := \frac{1 + \sqrt{3}}{2}, \qquad \beta := \frac{1 + \sqrt{5}}{2}.$$

   Find polynomials $f(x)$ and $g(x)$ with integer coefficients such that $f(\alpha) = g(\beta) = 0$. For one of α, β there is a monic polynomial with integer coefficients – determine which one.

   (b) If $\alpha = \sqrt{2} + \sqrt{5}$, find a polynomial $f(x)$ with integer coefficients such that $f(\alpha) = 0$.
   Remark: If α is a root of a monic polynomial with integer coefficients, it is known as an algebraic integer.

6. [VTRMC **2013 # 5**] Let $f(x) = x^5 - 5x^3 + 4x$. In each part (i)–(iv), prove or disprove that there exists a real number c for which $f(x) - c = 0$ has a root of multiplicity

   (i) one, (ii) two, (iii) three, (iv) four.

7. Suppose that $f(x)$ is a polynomial of degree n. It is a general principle that if f is uniquely determined if its value is known at $n+1$ points, say $f(a_j) = b_j$ for $0 \le j \le n$. The *Lagrangian Interpolation* formula then states that

$$f(x) = \frac{(x-a_1)(x-a_2)\cdots(x-a_n)}{(a_0-a_1)(a_0-a_2)\cdots(a_0-a_n)}b_0 + \frac{(x-a_0)(x-a_2)\cdots(x-a_n)}{(a_1-a_0)(a_1-a_2)\cdots(a_1-a_n)}b_1 + \cdots + \frac{(x-a_0)(x-a_1)\cdots(x-a_{n-1})}{(a_n-a_0)(a_n-a_1)\cdots(a_n-a_{n-1})}b_n.$$

   (a) Determine the quadratic polynomial with values $f(0) = 1, f(1) = 4, f(2) = 9$.

   (b) Show that the Lagrangian Interpolation formula works as claimed; i.e., that $f(a_j) = b_j$ for all $0 \le j \le n$.

   (c) Suppose that $f(x)$ satisfies $f(0) = f(1) = f(2) = 1$. The Lagrangian interpolation formula says that there is a unique quadratic polynomial $f(x) = ax^2 + bx + c$ with these values. But then $g(x) = f(x) - 1$ has **three** roots, at $x = 0, 1$, and 2 – and it is clear geometrically that a parabola can have at most two roots! Determine the coefficients of $f(x)$ and explain the apparent paradox.

8. (a) Suppose that $P(x)$ is a polynomial of degree n such that $P(1) = P(2) = \cdots = P(n) = 1$. Evaluate $P(n+1)$.

   (b) [Gelca-Andreescu **151**] Let $P(x)$ be a polynomial of degree n. Knowing that

$$P(k) = \frac{k}{k+1}, \quad k = 0, 1, \ldots, n,$$

   find $P(m)$ for $m > n$.

9. [Putnam **2003 B1**] Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

   holds identically?

10. (a) Neither of the following quartic polynomials have any rational roots. However, one of them factors into lower-degree polynomials with **integer** coefficients – determine which one:

$$x^4 + 1 \qquad \text{or} \qquad x^4 + x^2 + 1.$$

    (b) Factor each of the following quartic polynomials into lower-degree polynomials with **real** coefficients:

$$x^4 - 2; \qquad\qquad x^4 - x^2 + 1.$$

11. [Putnam **2003 B4**] Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z-r_1)(z-r_2)(z-r_3)(z-r_4)$, where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.