LSU Problem Solving Seminar - Fall 2018
Nov. 28: Putnam Review / Miscellaneous

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Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

Putnam Mathematical Competition, Sat., Dec. 1
Lockett Hall 232, 8:30 A.M. – 5:00 P.M.

Test-taking tips:

- **Format.** The Exam is given in two 3-hour sessions of 6 problems each, with a lunch break from 12:00 – 2:00 P.M. The morning session's problems are labeled **A1** – **A6**, and the afternoon's **B1** – **B6**.
- **Grading.** Each problem is graded out of **10** points, for a maximum possible score of 120. Typically there is very little partial credit given, and a submitted problem will receive **0, 1, 2, 9,** or **10** points.
- **A1/A2/B1.** In recent years these three problems have been the "easiest" part of the exam. More generally, the problems in each session are roughly ordered by difficulty. This is not an absolute rule, but you should expect that **A1** will have a relatively short solution, whereas **A6** may not. You should devote at least 15 minutes each to **A1, A2, B1** before moving on to the rest of the Exam.
- **1 hour per write-up.** In order to get full credit, your solutions must be written very carefully. If you use a result from a course, refer to it by name (e.g. Fundamental Theorem of Calculus). After you solve a problem, you should plan on spending approximately one hour writing your solution. In light of the grading described above, it is better to solve one problem completely than several problems partially.

Main Problems

This week's practice sheet provides a detailed look at several problems from previous Putnam Exams. Each Exam problem is **preceded** by a related problem that illustrates some relevant concepts in a simpler context.

1. (a) Let $f(x) = x^3 - 3x$. Calculate $f'(x)$, and determine its roots (the points such that $f'(x) = 0$). How do these roots compare to the roots of $f(x)$?

   (b) Rolle's Theorem states that if $f(x)$ is a differentiable function, and $f(a) = f(b)$, then there exists a point $a < c < b$ such that $f'(c) = 0$.

   Suppose that $f(x)$ has zeros at $x = a$ and $x = b$, with $a < b$. Apply Rolle's Theorem to $e^x f(x)$, and conclude that $f(c) + f'(c) = 0$ for some $a < c < b$. Check directly that this is true for $f(x) = x^2 - 1$.

2. [Putnam **2015 B1**] Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

3. Find the smallest positive integer with 7 or more distinct divisors.

4. [Putnam **1988 B1**] A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

5. For $n \geq 1$, define the matrices

$$A_n := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{pmatrix}.$$

For example,

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Calculate $\det(A_n)$.

Hint: For an inductive argument, subtract row $n-1$ from row n!

6. [Putnam **2014 A2**] Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

7. Suppose that every point in the plane is colored Red or Black. Show that there must exist two points of the same color that are exactly one unit apart.

Hint: Not all points can be Red (why?), so pick an arbitrary Black point P. Now consider all of the points that are exactly distance one from P....

8. [Putnam **1988 A4**]

   (a) If every point of the plane is colored one of three colors, do there necessarily exist two points of the same color exactly one unit apart?

   (b) What if "three" is replaced by "nine"?

   Justify your answers.

9. (a) Prove that for any positive real number x,

$$\sqrt{x} \leq \max\left\{2x, \frac{1}{2}\right\}.$$

   (b) Suppose that x is a positive real number. Show that for any positive integer n and positive constant c,

$$x^{1-\frac{1}{n}} \leq \max\left\{cx, \frac{1}{c^{n-1}}\right\}.$$

10. [Putnam **1988 B4**] Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} a_n^{n/(n+1)}$.