- Virginia Tech Mathematics Contest. Sat., Oct. 27. **Sign-up deadline: Sep. 28**.
- Putnam Mathematical Competition. Sat., Dec. 1. **Sign-up deadline: Oct. 5**.

LSU Problem Solving Seminar - Fall 2018
Sep. 5: Pigeonhole Principle and Invariants

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Useful facts and strategies:

- **Pigeonhole Principle.** If more than n objects are distributed among n sets, then some set contains multiple objects. **(Advanced Version.)** If more than nk objects are distributed among n sets, then one contains more than k objects.
- **Well-Ordering Property and Infinite Descent.** Every subset of natural numbers has a least element, or, equivalently, there is **no** infinite sequence of decreasing positive integers $n_1 > n_2 > n_3 > \cdots > 0$.

  This principle frequently applies to rational numbers as well; if r_i are rational numbers whose denominators are all **bounded**, then there is no infinite sequence $r_1 > r_2 > \cdots > 0$.
- **Invariants/Monovariants.** If you are asked about the possible outcomes of a procedure, try to find some *invariant* property that remains the same at each step. To show that a certain procedure ends in a **finite** number of steps, find a *monovariant:* a measurable quantity that is constantly increasing or decreasing.

Warm Up

1. (a) Assume that the majority of students at LSU have a First, Middle, and Last name, and correspondingly, three initials (e.g. Mike the "Fighting" Tiger has initials MFT). Show that there are at least two students at LSU* who have the same three initials.

   * There are over 35,000 students at LSU.

   (b) There are 64 Parishes in Louisiana, and 75% of LSU students are from the state. Show that there are at least two students at LSU who are from the same Parish, whose last names start with the same letter, and whose birthdays are in the same month.

2. The integers $1, 2, 3, \ldots, 2018$ are written on a blackboard. Choose any two adjacent numbers a and b, erase them, and write the single number $ab + a + b$ in their place. Now apply this same rule to the resulting list of integers.

   (a) How many steps are required until just a single number remains?

(b) Show that the final number is the same no matter which sequence of pairs are chosen. What is the final number?

Hint: Begin by considering smaller examples: One possible sequence of steps for $1, 2, 3, 4$ *would be:*

$$1 \quad \underbrace{2 \quad 3}_{2\cdot 3+2+3=11} \quad 4$$

$$\underbrace{1 \quad 11}_{1\cdot 11+1+11=23} \quad 4$$

$$\underbrace{23 \quad 4}_{23\cdot 4+23+4=119}$$

$$119$$

3. You are allowed to make a sequence of *Knight's moves* in the plane: a distance of 2 in one axis direction, followed by a distance of 1 in the other axis direction. For example, from the origin $(0, 0)$, there are eight possible moves, to

$$(2, 1),\ (2, -1),\ (1, 2),\ (1, -2),\ (-1, 2),\ (-1, -2), (-2, 1),\ (-2, -1).$$

(a) Starting from $(0, 0)$, is it possible to reach the point $(3, 5)$?

(b) What is the minimum number of moves needed to reach the point $(5, 3)$?

(c) Is it possible to reach **every** integer point in the plane? Prove your answer.

Hint: Is it possible to reach $(1, 0)$*?*

Main Problems

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4. (a) Suppose that 5 points are placed in a square with side length 1. Show that there are two points at a distance of most $\frac{1}{\sqrt{2}}$.

(b) Now suppose that 9 points are placed in a triangle of area 1. Show that there are 3 points that form a triangle of area at most $\frac{1}{4}$.

(c) Finally, show that if 5 points are placed in a triangle with area 1, then there is a triangle with area at least $\frac{1}{6}$ that does not contain any of the points in the interior.

5. A popular style of arithmetic puzzle is a *Cryptarithm*, where each letter is replaced by a distinct digit (0 – 9) so that the resulting expression is true. Each of the following puzzles has a **unique** solution; solve them systematically, without guessing:

(a)
$$\begin{array}{cccccccc} & & P & U & T & N & A & M \\ + & & P & U & T & N & A & M \\ \hline & S & T & U & D & E & N & T \end{array}$$

(b)
$$\begin{array}{cccccccc} & & N & U & T & T & E & R \\ & & B & U & T & T & E & R \\ + & & & & O & R & E & O \\ \hline & P & R & O & B & L & E & M \end{array}$$

Hint: For (a), start by looking at the possible choices for S and U. . . .

6. (a) A different Knight sets out to slay another magical Dragon, who has 3 fire-breathing Heads and 3 poisonous Tails. Depending on his luck, in a single blow the Knight can kill 1 or 2 Heads, or 1 or 2 Tails. However, the Dragon immediately regenerates as follows:

$$\begin{array}{rlrl} 1 \text{ Head cut} & \to 1 \text{ Head grows;} & 2 \text{ Heads cut} & \to \text{ nothing grows;} \\ 1 \text{ Tail cut} & \to 2 \text{ Tails grow;} & 2 \text{ Tails cut} & \to \text{ 1 Head grows.} \end{array}$$

Show that it is possible for the Knight to slay the Dragon by cutting off all of the Heads and Tails. What is the minimum number of blows?

(b) Now suppose that the Knight must face the fearsome General Dragon, who has A Heads and B Tails. Find the minimum number of blows needed to defeat this beast.

Hint: Is this related to Problem 3?

7. (a) If $M \geq 1$ is an integer, prove that

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{M(M+1)} = 1 - \frac{1}{M+1}.$$

(b) Evaluate

$$\sum_{j=10}^{\infty} \frac{1}{j(j+1)} = \frac{1}{10 \cdot 11} + \frac{1}{11 \cdot 12} + \frac{1}{12 \cdot 13} + \cdots .$$

(c) Evaluate

$$\sum_{j=10}^{\infty} \frac{1}{j(j+1)(j+2)} = \frac{1}{10 \cdot 11 \cdot 12} + \frac{1}{11 \cdot 12 \cdot 13} + \cdots .$$

8. [VTRMC **1986 # 8**] Find all pairs N, M of positive integers, $N < M$, such that

$$\sum_{j=N}^{M} \frac{1}{j(j+1)} = \frac{1}{10}.$$

9. [Putnam **1991 B1**] For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?