- Virginia Tech Mathematics Contest. Sat., Oct. 27. **Sign-up deadline: Sep. 28**.
- Putnam Mathematical Competition. Sat., Dec. 1. **Sign-up deadline: Oct. 5**.

LSU Problem Solving Seminar - Fall 2018

Sep. 12: Calculus and Functions

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Useful facts and strategies:

- **Intermediate Value Theorem.** Suppose that $f(x)$ is a continuous function defined on the interval $[a,b]$, and r is a value in between $f(a)$ and $f(b)$, so that

$$f(a) < r < f(b) \quad \text{or} \quad f(a) > r > f(b).$$

  Then there is some point c in the interior of the interval, $a < c < b$, such that $f(c) = r$.

  In other words, a continuous function cannot "skip" any values.

- **Differentiability.** A function f is differentiable at a if the following limit exists:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

  If so, this value is denoted by $f'(a)$.

- **Mean Value Theorem.** Suppose that $f(x)$ is differentiable on the interval $[a,b]$. Then there is a point $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  In other words, a differentiable function must achieve its **average** *slope at some point.*

- **Critical Points.** If $f(x)$ is a differentiable function on an interval $[a,b]$, then its maxima/minima must occur at the end points or the **critical points**, where are those x such that $f'(x) = 0$. The maxima/minima are classified by the negativity/positivity of $f''(x)$.

  Something similar is true for multivariable functions; the maxima/minima of $f(x,y)$ also occur when $\frac{\partial}{\partial x} f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are zero, but there is the additional possibility of a *saddle point*.

- **L'hospital's Rule.** Suppose $f(x)$ and $g(x)$ are differentiable. If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an **indeterminate form** (i.e., $\frac{0}{0}$ or $\frac{\infty}{\infty}$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

- **Continuity and Limits.** If g is a continuous function (or even just has a limit that exists at $f(a)$), then

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

- **Taylor Series.** The Taylor series of $f(x)$ around $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

  In general this series will hold for values of x in some interval around a.

Warm Up

1. Yesterday the high temperature was 86 degrees (Fahrenheit), and the low was 73 degrees. If the high today is 91, and the low is 71, show that there is some time of day where the temperature yesterday and today were identical!

   Hint: Let $f(t)$ be the temperature yesterday at time t, and let $g(t)$ be the temperature today at time t. If $t_{\max}$ is the time such that $g(t_{\max}) = 91$, what can you say about $g - f$?

2. Suppose that a (circular) pizza has exactly 24 pieces of pepperoni (all of which are identical).

   (a) If the pizza is cut into 8 slices, show that one piece has at least 3 pieces of pepperoni. Note that some pieces of pepperoni may have been cut, so the number of pieces in a slice includes the sum of the precise fractional parts; for example, a slice of pizza may include 2 whole pieces of pepperoni, one half of another piece, and 30% of another piece, for a total of $2 + 0.5 + 0.3 = 2.8$ pieces of pepperoni.

   (b) Now show that there is a way to cut the pizza into 8 slices of **equal** area such that one piece contains **exactly** 3 pepperoni.

   Hint: Start by cutting into 8 slices of equal area starting from an arbitrary location. If there is not a slice with exactly 3 pepperoni, show that there must be two adjacent slices such that one has more than 3 pepperoni, and one has less than 3. What happens if you rotate the position of the cuts?

3. A cyclist completes a 10 mile ride in 40 minutes, for an average speed of 4 minutes per mile.

   (a) Show that there must have been a two-mile segment of the ride that was completed in **exactly** 8 minutes.

   (b) Is it necessarily true that there was also a three-mile segment that matched the average speed (i.e., that took **exactly** 12 minutes)?

Main Problems

4. (a) Find the Taylor series around $x = 0$ for $f(x) = \sqrt{1+x}$.

   (b) Are there any k such that the Taylor series of $f(x) = \sqrt{1+kx}$ around $x = 0$ has integer coefficients? If so, find **all** such k.

5. (a) Let

   $$f(x) = e^x, \qquad \text{and} \qquad g(x) = ax + b,$$

   an arbitrary linear function. Prove that the graphs of f and g intersect in at most two points.

   Hint: Let $h(x) := g(x) - f(x)$ and consider the critical and inflection point(s) of h. What does this tell you about the zeroes of h?

   (b) Give an exact characterization for all possible behaviors:

- For which a and b are there 0 intersection points?
- For which a and b is there 1 intersection point?
- For which a and b are there 2 intersection points?

6. Find all real solutions to
$$2^{x+1} + 2x = 3^x + 3,$$
and prove that your list is complete.

7. [**Gelca-Andreescu 457**] Find all real numbers x and y that are solutions to the system of equations
$$\begin{aligned} 3^x - 3^y &= 2^y \\ 9^x - 6^y &= 19^y. \end{aligned}$$

8. (a) Suppose that (x_1, y_1) and (x_2, y_2) are two points such that $y_1, y_2 > 0$. Find a formula for the length of the shortest path that goes from (x_1, y_1) to (x_2, y_2) and touches the x-axis.

   Hint: You should **reflect** *on this problem. . . .*

   (b) [VTRMC **1992 # 2**] Assume that $x_1 > y_1 > 0$ and $y_2 > x_2 > 0$. Find a formula for the shortest length ℓ of a planar path that goes from (x_1, y_1) to (x_2, y_2) and that touches both the x-axis and the y-axis. Justify your answer.

9. [Putnam **1972 B1**] Show that the power series representation of
$$f(x) = \sum_{n=0}^{\infty} \frac{x^n (x-1)^{2n}}{n!} = \sum_{m=0}^{\infty} a_m x^m$$
cannot have three successive zero coefficients.

10. (a) Find all continuous functions on $[0, \infty)$ such that
    - $f(x) = f\left(\dfrac{x^2}{x+1}\right)$,
    - $f(0) = 1$.

    Hint: What happens if you iterate the relation? For example, $f(1) = f(\frac{1}{2})$*, and then* $f(\frac{1}{2}) = f\left(\dfrac{\frac{1}{4}}{\frac{1}{2}+1}\right) = f(\frac{1}{6})$, ...

    (b) If n is a positive integer, evaluate
$$\lim_{x \to 1^-} \frac{\sqrt{1-x^n}}{\sqrt{1-x}}.$$

11. [Putnam **2012 A3**] Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that

    (i) $f(x) = \dfrac{2-x^2}{2} f\left(\dfrac{x^2}{2-x^2}\right)$ for every x in $[-1, 1]$,

    (ii) $f(0) = 1$, and

    (iii) $\displaystyle\lim_{x \to 1^-} \frac{f(x)}{\sqrt{1-x}}$ exists and is finite.