- Virginia Tech Mathematics Contest. Sat., Oct. 27. **Sign-up deadline: Sep. 28**.
- Putnam Mathematical Competition. Sat., Dec. 1. **Sign-up deadline: Oct. 5**.

LSU Problem Solving Seminar - Fall 2018
Sep. 19: Enumeration

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Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

Useful facts and strategies: (n and k are non-negative integers)

- **Permutations.** The number of ordered lists of k distinct elements chosen from a set of n objects is $P(n,k) := \dfrac{n!}{(n-k)!}$.
- **Binomial Coefficients.** Given two non-negative integers n and k, the number of ways of choosing k (unordered) objects from a set of n is $\dbinom{n}{k} := \dfrac{n!}{k!(n-k)!}$ (this is read as "n choose k"). They satisfy the recurrence $\dbinom{n}{k} = \dbinom{n-1}{k} + \dbinom{n-1}{k-1}$.
- **Binomial Theorem.** For an integer $n \geq 0$, $(1+x)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k$.
- **Number of subsets.** There are 2^n distinct subsets of a set with n elements.
- **Inclusion-Exclusion.** Suppose that $A_1, A_2, \ldots, A_n$ are sets. Then

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$
$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|,$$

and in general, with $A_I := \bigcap_{i \in I} A_i$,

$$|A_1 \cup \cdots \cup A_n| = \sum_{I \subseteq [1,n]} (-1)^{|I|} A_I.$$

Warm Up

1. A Personal Identification Number (PIN) is a sequence of 4 digits from $\{0, 1, \cdots, 9\}$.
   (a) How many possible PINs are there?
   (b) How many PINs begin with 9?
   (c) How many PINs do **not** contain a 9?
   (d) How many PINs contain at least one 9?
2. A made-to-order salad restaurant offers a number of toppings, (*Vegetables*): Avocado, Beets, Green Beans, Herbs, Iceberg Lettuce, and Jicama; (*Meats/Proteins*): Chicken, Diced Ham, Eggs, and Fish. How many distinct choices are there for each of the following menu options?

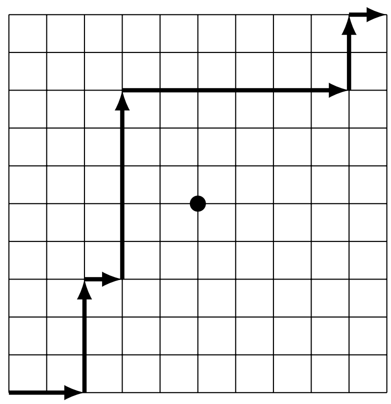
(a) A basic *Salad* allows the choice of 3 distinct vegetables and 1 type of meat. Note that as the toppings are mixed in the salad, the order doesn't matter.

(b) A *Green Smoothie* contains 4 portions of vegetables blended with greens, where the same vegetable may be chosen multiple times.

(c) A *Meat-Lovers Flatbread* consists of 3 portions of meat that are layered and roasted; the same meat may be chosen multiple times, and since the top items drip onto the lower, the order matters.

(d) The *Small-Plate Sampler* consists of 4 items served with different dressings and dips. The items may be vegetables or meats, but each must be distinct. Furthermore, the order matters, as each dressing is different.

3. Consider *lattice paths* from $(0, 0)$ to $(10, 10)$; this means that each step must be some number of units to the right or up.



(a) How many lattice paths are there from $(0, 0)$ to $(10, 10)$?

(b) How many of those lattice paths pass through the point $(5, 5)$?

(c) Finally, consider only the paths that do **not** pass through the point $(5, 5)$. Each of these paths instead passes *above* or *below* the point; for example, the path in the figure passes above. Determine the proportion of these paths that pass above, and the proportion that pass below.

Main Problems

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4. In this problem you will show that you are able to count to 10000. Any 4-digit PIN has a *repetition type* that describes how many times each different digit occurs. For example, the PIN 8729 has repetition type $(1, 1, 1, 1)$, since each digit occurs once, while 3943 has type $(2, 1, 1)$, since 3 occurs twice, and 9 and 4 each occur once. As one more example, 7777 has type (4), since the digit 7 occurs four times (and there are no other digits).

Now group the PINs by repetition type and count the size of each group. For example, if a PIN has repetition type (4), then it consists of a single digit repeated four times, and there are only **10** possibilities, one for each digit. Determine the number of PINs for the other repetition types, and make sure that they sum to 10000!

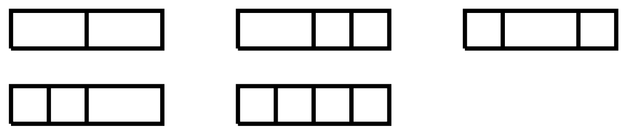
5. The *Fibonacci sequence* are defined by $F_0 = F_1 = 1$, and $F_{n+1} := F_n + F_{n-1}$ for $n \geq 1$.

   (a) Calculate the first several terms in the sequence.

   (b) Find (and prove) a formula for the sum of the even-indexed terms,

   $$F_0 + F_2 + \cdots + F_{2n} = ?$$

   (c) Let $d(n)$ denote the number of *domino tilings* of length n, which are composed of 2×1 dominoes and 1×1 squares. For example, for $n = 4$ the tilings are

   so $d(4) = 5$. Find and prove an expression for $d(n)$ in terms of the Fibonacci numbers.

   (d) Now give a *combinatorial* proof of the identity from part (b) by counting the number of domino tilings of length $2n + 1$ in two different ways. First, use the formula from part (c). Second, consider the initial run of dominoes in such a tiling. For example, if the tiling begins with no dominoes, then it must begin with a square, leaving a length of $2n$ to be tiled. If the tiling begins with exactly one domino, then the following space is a square, leaving a length of $2n - 2$; continue this argument to cover all possible cases.

6. [**Gelca-Andreescu 962**] For a positive integer $n, n \geq 3$, consider the points $A_1, A_2, \cdots, A_n$ on a circle in this order, and place the numbers $1, 2, \cdots, n$ randomly at these points.

   (a) Show that the sum of the absolute values of the differences of neighboring numbers is greater than or equal to $2n - 2$.

   (b) For how many arrangements of these numbers is the sum **exactly** $2n - 2$?

7. (a) [VTRMC **1979 # 8**] There are $2n$ balls in the plane such that no three balls are on the same line and such that no two balls touch each other. n balls are red and the other n balls are green. Show that there is at least one way to draw n line segments by connecting each ball to a unique different colored ball so that no two line segments intersect.

   (b) Find a configuration of 10 balls such that there is only **one** possible way to draw 5 nonintersecting lines.

8. Suppose that a necklace contains $2n$ beads, where n of the beads are labeled $+1$ and n of the beads are labeled -1. Show that it is possible to cut the necklace to form a string such that **all** of the initial substrings have a nonnegative sum. This means that the sum of the first k beads must be nonnegative for all $1 \leq k \leq 2n$ (which immediately means that the first bead must be $+1$).

   Remark: The number of such strings of length $2n$ *is known as the* Catalan number C_n.

9. [Putnam **1995 A4**] Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \cdots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \qquad \text{for} \quad k = 1, 2, \cdots, n.$$