- Virginia Tech Mathematics Contest. Sat., Oct. 27. **Sign-up deadline: Sep. 28**.
- Putnam Mathematical Competition. Sat., Dec. 1. **Sign-up deadline: Oct. 5**.

LSU Problem Solving Seminar - Fall 2018

Sep. 26: Geometry and Trigonometry

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Useful facts:

- **Triangle Inequality.** If a, b, c are the side lengths of a triangle, then $a < b + c$.
- **Pythagorean Theorem.** Suppose that ABC is a right triangle, with $\angle ABC = 90°$. If the (opposing) side lengths are $|\overline{AB}| = c$, $|\overline{AC}| = b$, $|\overline{BC}| = a$, then $b^2 = a^2 + c^2$.
- **Law of Cosines.** If a triangle has sides of lengths a, b, and c, and α is the angle opposite the side of length a, then
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$
- **Law of Sines.** If β is the angle opposite b, and γ is the angle opposite c, then
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} = \frac{1}{2R},$$
where R is the radius of the circumscribed circle (which contains the vertices of the triangle).
- **Pythagorean Formula.** For all x,
$$\sin^2(x) + \cos^2(x) = 1,$$
- **Addition Formulas.** For all x and y,
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$
- **Heron's Formula.** If a triangle has side lengths a, b, and c, then its area is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s := \frac{a+b+c}{2}$ is the *semiperimeter.*
- **Prisms.** The volume of a prism of height h and base area A is $V = \frac{hA}{3}$.
- **Pick's Theorem.** Suppose that a polygon with integer vertices contains n integer points in its interior, and m integer points along its edges (including vertices). Then the area is $A = n + \frac{m}{2} - 1$.

Warm Up

1. (a) Prove that $\cos\alpha\cos\beta = \frac{1}{2}\big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\big)$.

(b) Prove that $\sin\alpha\sin\beta = -\frac{1}{2}\big(\cos(\alpha+\beta) - \cos(\alpha-\beta)\big)$.

2. (a) Prove that *half-angle* formula for the sine function, $\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{2}}$.

   (b) Evaluate $\sin\left(\frac{\pi}{8}\right)$.

3. (a) What is the area of the triangle with vertices $(0,0),(6,0)$ and $(5,3)$?

   (b) What is the area of the triangle with vertices $(0,0),(6,0)$ and $(1005,3)$?

   (c) Let T be the triangle with vertices $(0,0),(2,6)$ and $(5,3)$. Calculate the area of T.

Main Problems

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4. If a,b,c,d are positive real numbers, what is the area of the triangle with vertices $(0,0),(a,b)$ and (c,d)?

   Once you have obtained a formula, try to prove it in more than one way! Some possible approaches include:

   (*i*) Rectangles and Right Triangles;

   (*ii*) Vector dot product and/or cross product;

   (*iii*) Properties of Determinants and Areas/Volumes.

5. (a) Verify that Pick's Theorem (see the list of Useful Facts!) holds for the triangle T from Problem 3 (c).

   (b) What is the area of the (non-convex) polygon whose boundary consists of line segments connecting the following points, in order:

$$(3,1),\ (4,3),\ (6,2),\ (5,4),\ (6,7),\ (3,4),\ (3,1).$$

   (c) Prove Pick's Theorem for any triangle with vertices $(0,0),(1,0),(a,b)$, where a and b are positive integers.

   Hint: Draw another triangle with vertices $(a,b),(a+1,b),(1,0)$, and use a symmetry argument ....

6. (a) Prove that for any positive integer $n \geq 1$ and real x,

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(\left(n+\frac{1}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)}.$$

   Hint: Multiply the $\sin\left(\frac{x}{2}\right)$ to the left-hand side, and induct on n, using Problem 1 (b).

   (b) Prove that for $n \geq 1$,

$$\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \cdots + \sin\left(\frac{(n-1)\pi}{n}\right) = \cot\left(\frac{\pi}{2n}\right).$$

   Remark: An algebraic proof follows directly from part (a). However, there is also a beautiful geometric proof that is based on constructing a right triangle with angle $\frac{\pi}{2n}$ and opposite leg of length 1....

7. [**Gelca-Andreescu 13**] Prove that $|\sin nx| \leq n|\sin x|$ for any real number x and positive integer n.

8. [VTRMC **1985 # 4**] Consider an infinite sequence $\{C_k\}_{k=0}^{\infty}$ of circles. The largest, C_0, is centered at $(1,1)$ and is tangent to both the x and y-axes. Each smaller circle C_n is centered on the line through $(1,1)$ and $(2,0)$, and is tangent to the next larger circle and to the x-axis. Denote the diameter of C_n by d_n for $n = 0, 1, 2, \ldots$.. Find

   (a) d_1,

   (b) $\displaystyle\sum_{n=0}^{\infty} d_n$.

9. [Putnam **1986 B1**] Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?