LSU Problem Solving Seminar - Fall 2018
Oct. 3: Sequences and Series

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Useful facts:

- **Limit of a Sequence.** A sequence $\{a_n\}_{n=1}^{\infty}$ *converges to a limit* ℓ if for any $\varepsilon > 0$ there is an N such that $|a_n - \ell| < \varepsilon$ for all $n > N$.
- **Geometric Series.** If $|x| < 1$, then $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$.
- **Ratio Test.** Let $L := \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$. If $L < 1$, then $\sum_{n\geq 1} a_n$ converges, and if $L > 1$, then the sum diverges. If $L = 1$, the test is inconclusive.
- **Monotone Convergence.** If $a_1 \leq a_2 \leq \dots$ and all $a_n \leq B$ for some constant B, then $\lim_{n\to\infty} a_n$ exists (though it may be less than B).
- **Alternating Series.** If $a_1 \geq a_2 \geq \dots$ and $\lim_{n\to\infty} a_n = 0$, then the alternating series $a_1 - a_2 + a_3 - a_4 + \dots$ converges.
- **Integral Comparison.** If $f(x)$ is a decreasing function for $x \geq 0$, then $\sum_{n\geq 1} f(n) \leq \int_0^\infty f(x)dx$.
- **Linear Recurrences.** The *characteristic polynomial* associated to a (homogeneous) recurrence $a_{n+k} = c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n$ is $p(x) := x^k - c_{k-1}x^{k-1} - \cdots - c_1 x - c_0$. If $p(x)$ has **distinct** roots $\lambda_1, \dots, \lambda_k$, then the general solution to the recurrence is

$$a_n = b_1\lambda_1^n + \cdots + b_k\lambda_k^n,$$

where the constants are determined by k initial values. If there is a **repeated** root λ of order m, then the general solution has the term $(d_{k-1}n^{k-1} + \cdots + d_1 n + d_0)\lambda^n$.

Warm Up

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1. (a) Evaluate $S_1 = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \cdots$.

   (b) Evaluate $S_2 = \dfrac{1}{2} - \dfrac{1}{4} + \dfrac{1}{8} - \dfrac{1}{16} + \cdots$.

   Hint: How does $2 \cdot S_2$ compare to S_2?

2. Simplify the following expressions to rational fractions of the form $\frac{a}{b}$:

$$\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1}}}, \qquad \cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1}}}}, \qquad \cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1}}}}}.$$

3. (a) Evaluate $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$.

   Hint: What is the Taylor series for the function $\ln(1+x)$ *around* $x = 0$*?*

   (b) Evaluate $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \cdots$..

Main Problems

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4. (a) Let $q_1 = 1, q_2 = 3$, and for $n \geq 3$,

$$q_n = q_{n-1} + 2q_{n-2}.$$

   Find a general formula for q_n.

   (b) Let $p_1 = 2, p_2 = 2$, and for $n \geq 3$,

$$p_n = p_{n-1} + 2p_{n-2}.$$

   Find a general formula for p_n.

   (c) How do these series relate to Problem 2? Can you conjecture and/or prove any interesting properties?

   For example, what can you say about $p_n - q_n$? What is $\lim_{n\to\infty} \frac{p_n}{q_n}$?

5. For a real number a, define an *infinite continued fraction* by

$$x_a := \cfrac{a}{1 + \cfrac{a}{1 + \cfrac{a}{1 + \cfrac{a}{1 + \ddots}}}}.$$

   (a) Assuming that the limit exists, determine the value of x_a.

   Hint: Note that $x_a = \frac{a}{1 + x_a}$.

   (b) Recall Problems 2 and 4. What is the value of x_2?

   (c) What is the value of

$$\cfrac{3}{4 + \cfrac{3}{1 + \cfrac{3}{4 + \cfrac{3}{1 + \cfrac{3}{4 + \ddots}}}}}\ ?$$

   Hint: How does this relate to x_a*? Try* $a = \frac{3}{4}$*!*

6. [**Gelca-Andreescu 300**] The sequence $a_0, a_1, a_2, \ldots$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right),$$

for all nonnegative integers m and n with $m \geq n$. If $a_1 = 1$, determine a_n.

7. (a) Evaluate $\displaystyle\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2}$.

   (b) Prove the integral comparison

$$\int_0^n \sqrt{x}\,dx \leq \left(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}\right) \leq \int_1^{n+1} \sqrt{x}\,dx.$$

   (c) Find the value of c such that $\displaystyle\lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n^c}$ exists, and then evaluate the limit.

8. (a) Use Integration by parts to find an antiderivative of $f(x) = x\log x$, i.e., evaluate $\int x\log x\,dx$.

   Remark: In higher mathematics, $\log x$ *typically means the natural logarithm* $\ln x$*. This is because the logarithm base b only differs from* $\ln x$ *by a constant (as* $\log_b x = \frac{1}{\ln b}\ln x$*), and* $\ln$ *has the "nicest" analytic properties.*

   (b) [VTRMC **1992 # 7**] Find $\displaystyle\lim_{n\to\infty} \frac{2\log 2 + 3\log 3 + \cdots + n\log n}{n^2\log n}$.

9. [Putnam **1999 A3**] Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$.