LSU Problem Solving Seminar - Fall 2018
Oct. 10: Integration

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Useful facts:

- **Partial Fractions.** If $f(x)$ is a polynomial whose degree is less than n, then there are constants $a_1, \ldots, a_n$ such that

$$\frac{f(x)}{(x-r_1)\cdots(x-r_n)} = \frac{a_1}{x-r_1} + \cdots + \frac{a_n}{x-r_n}.$$

  (Here the roots r_i must be distinct – there is a more complicated version for repeated roots.)

- **Fundamental Theorem(s) of Calculus.** Suppose that $f(x)$ is a continuous function.
  - If $F(x)$ is an antiderivative of f, then $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$.
  - Define $g(x) := \displaystyle\int_a^x f(t)dt$. Then $g'(x) = f(x)$.

- **Integration By Parts.** Suppose that f and g are differentiable. Then

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx.$$

- **Substitution.**

$$\int_{x=a}^{b} f(u(x))u'(x)dx = \int_{u=u(a)}^{u(b)} f(u)du.$$

- **Symmetries and Substitution.** Always remember that integration problems on Mathematics Contests are meant to have solutions! A complicated integral often has a hidden symmetry or substitution that makes it much easier to evaluate. For example, if $f(x)$ is an *odd* function, then $\int_{-a}^{a} f(x)\, dx = 0$.

Warm Up

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1. Calculate the following antiderivatives (indefinite integrals):

   (a) $\displaystyle\int x^2 \cos\left(x^3\right)\, dx$

   (b) $\displaystyle\int \frac{x}{2x^2+3}\, dx$,

   (c) $\displaystyle\int \frac{1}{x^2+4x+5}\, dx$,

2. Evaluate the following (definite) integrals with as **little** computation as possible:

(a) $\displaystyle\int_0^3 (6-2x)\,dx.$

Hint: Think geometrically.

(b) $\displaystyle\int_{-\pi}^{\pi} \sin x\,dx.$

(c) $\displaystyle\int_{-\pi}^{\pi} \sin^2 x\,dx.$

Hint: Recall that $\sin^2 x + \cos^2 x = 1$.

3. Use Integration by parts to evaluate

$$\int_0^\infty t^2 e^{-t}\,dt.$$

Main Problems

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4. (a) Evaluate

$$\int_{-2}^{2} \frac{\sqrt{3+x^3}}{\sqrt{3-x^3}+\sqrt{3+x^3}}\,dx.$$

Hint: Are there any symmetries in the integrand? What happens if you replace $x \mapsto -x$*?*

(b) Evaluate the following integral – this can be done **without** finding an antiderivative!

$$\int_0^2 \frac{x\sqrt{1+x^2}}{\sqrt{1+x^2}+\sqrt{5-x^2}}\,dx = ?$$

5. (a) [**Gelca-Andreescu 540**] Compute the integral

$$\int \frac{x^2+1}{x^4-x^2+1}\,dx.$$

Hint: For a first approach, write the integrand as $\dfrac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}$. *Now make the change of variables* $t = x - \frac{1}{x}\ldots$

(b) Now find the antiderivative using partial fractions: First, observe that $x^4-x^2+1 = \left(x^2+\sqrt{3}x+1\right)\left(x^2-\sqrt{3}x+1\right)$. Then find constants such that

$$\frac{x^2+1}{x^4-x^2+1} = \frac{ax+b}{x^2+\sqrt{3}x+1} + \frac{cx+d}{x^2-\sqrt{3}x+1}.$$

Now evaluate the antiderivatives using the arctangent.

Remark: These two approaches illustrate the importance of the integration constant "+C", as the two solutions actually differ by $\frac{\pi}{2}$*, though of course they have the same derivative! If you are ambitious, the precise relation between the two forms can be shown using the summation formula* $\arctan(u)+\arctan(v) = \arctan\left(\frac{u+v}{1-uv}\right)$ *and inverse formula* $\arctan(u)+\arctan(u^{-1}) = \frac{\pi}{2}$.

6. (a) Evaluate $I_1 = \int_0^1 \ln x \, dx$.

   Hint: Write $\ln x = 1 \cdot \ln x$*, and use Integration by parts, where* $f'(x) = 1$ *and* $g(x) = \ln x$*.*

   (b) Evaluate $I_2 = \lim_{r\to 0} \left(\int_0^1 x^r \, dx \right)^{\frac{1}{r}}$.

   Hint: Recall that $e = \lim\limits_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.

   (c) To check your work, your answers should satisfy $e^{I_1} = I_2$.

7. [VTRMC **1986 # 5**] Verify that, for $f(x) = x + 1$,

$$\lim_{r\to 0^+} \left(\int_0^1 (f(x))^r \, dx \right)^{\frac{1}{r}} = e^{\int_0^1 \ln f(x)\, dx}.$$

8. [Putnam **1987 B1**] Evaluate

$$\int_2^4 \frac{(\ln(9-x))^{1/2}}{(\ln(9-x))^{1/2} + (\ln(x+3))^{1/2}} \, dx.$$

9. (a) Recall the rules of differentiation for integrals:

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x);$$
$$\frac{d}{dx} \int_a^b f(t,x)\, dt = \int_a^b \frac{\partial}{\partial x} f(t,x)\, dt.$$

   Furthermore, the Chain Rule for two-variable functions states that

$$\frac{d}{dx} f(u(x), v(x)) = \frac{\partial f}{\partial u} \cdot \frac{du}{dx} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dx}.$$

   Give a formula for the following derivative:

$$\frac{d}{dx} \int_a^x f(t,x)\, dt.$$

   (b) Let

$$I(y) := \int_0^y \left(y - 2x^2\right) dx.$$

   Find the maximum value of $I(y)$.

   (c) Show that if a and b are nonnegative reals, $\sqrt{a^2+b^2} \le a + b$.

10. [Putnam **1991 A5**] Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 \le y \le 1$.