LSU Problem Solving Seminar - Fall 2018
Oct. 17: Number Theory

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

Useful facts:

- **Divisibility Tests.** A positive integer n is divisible by:
  - 2 if its last digit is a multiple of 2;
  - 3 if the sum of its digits is a multiple of 3;
  - 4 if its last two digits are a multiple of 4;
  - 5 if its last digit is 0 or 5;
  - 7 if ____
  - 9 if the sum of its digits is a multiple of 9;
  - 11 if the alternating (with plus and minus signs) sum of its digits is a multiple of 11.
- The current year has the prime factorization $2018 = 2 \cdot 1009$, and the previous year, 2017, is prime.
- **Fermat's Little Theorem.** If p is prime and a is any integer, then $a^p - a$ is a multiple of p.
- **Remainders of Squares.** For any integer n, n^2 can only have the following remainders:

  0 or 1 when divided by 3; 0 or 1 when divided by 4; 0, 1, 4, 5, 6, or 9 when divided by 10.
- **Greatest Common Divisors / Least Common Multiples.** If a and b are integers with prime factorizations $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, b = p_1^{\beta_1} \cdots p_r^{\beta_r}$, their greatest common divisor is

  $$\gcd(a,b) = p_1^{\min\{\alpha_1,\beta_1\}} \cdots p_r^{\min\{\alpha_r,\beta_r\}}.$$

  The least common multiple is found by replacing min by max.

  The equation $ax + by = N$ has integer solutions if and only if $\gcd(a,b)$ divides N.
- **Euler's Totient Function.** If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then define

  $$\phi(n) := n \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

  Among the integers $1, 2, \ldots, n$, exactly $\phi(n)$ of them satisfy $\gcd(a,n) = 1$. Furthermore, if $\gcd(a,n) = 1$, then $a^{\phi(n)}$ has remainder 1 when divided by n.
- **Casting Out Nines.** If n is an integer and s is the sum of its decimal digits, then $n - s$ is a multiple of 9.

Warm Up

---

1. Find the unique decimal solution to

   $$BA \times CA = DDD.$$

   In other words, find two 2-digit numbers that have the same ones digit such that their product is a three-digit number with a single digit repeated three times.

   Hint: The desired product is $D \cdot 111$ – can you factorize this further?

2. (a) Consider the integer
$$1234\_\_6.$$
Fill in the missing digit so that the integer is a multiple of 9. Then fill it in so that the integer is a multiple of 11.

   (b) One of the following expressions is a multiple of 11 – determine which one:
$$7^7 + 9^9 \qquad \text{or} \qquad 7^9 + 9^7\ ?$$

3. (a) Find positive integers a and b such that $a^2 - b^2 = 20$.

   (b) Find positive integers a and b such that $a^2 - b^2 = 45$.

   Remark: Both (a) and (b) have multiple solutions – try to find them all!

   (c) Are there positive integers a and b such that $a^2 - b^2 = 2018$?

Main Problems

---

4. Give a precise (and simple!) characterization of the integers n that can be expressed as the difference of two squares, i.e., $a^2 - b^2 = n$.

5. (a) Find the prime factorization of $77, 777$, and 7777.

   (b) Prove that every prime $p \neq 2, 5$ occurs in the prime factorization of some integer of the form $77 \cdots 7$.

6. Only one of the following equations has an integer solution – determine which one.
$$x^2 + 5y^4 = 2018;$$
$$x^2 + 2018y^3 = -15.$$

   Hint: Consider the remainders when dividing by 5. . . .

7. (a) Suppose that n is an integer. Is it possible that
$$(n+1)^2 = n^2 + (n-1)^2?$$
If so, find **all** solutions.

   (b) [VTRMC **1984 # 2**] Consider any three consecutive integers. Prove that the cube of the largest cannot be the sum of the other two.

8. (a) [VTRMC **1984 # 1**] Find the units digit in the sum $\displaystyle\sum_{k=1}^{99} k!$.

   (b) Find the first two exponents n such that 29^n ends with the digits 61.

9. (a) Suppose that a and b are both powers of 2. Show that at least one of a^2 or b^2 divides ab.

(b) Now suppose that a and b may (individually) be a power of 2 or 3. Find an example where neither a^2 nor b^2 divides ab. However, show that if a, b, and c are powers of 2 or 3, then at least one of a^2, b^2 or c^2 divides abc.

(c) [Adapted from **Gelca-Andreescu 10**] Let $P = \{p_1, p_2, \cdots, p_k\}$ be a set of distinct prime numbers. Suppose that $a_1, a_2, \cdots, a_{k+1}$ are positive integers whose factorization only includes primes in P. Let $A := a_1 \cdot a_2 \cdots a_{k+1}$. Must it be true that a_j^2 divides A for some j? Prove your answer.

10. [Putnam **2017 B2**] Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a + 1) + (a + 2) + \cdots + (a + k - 1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

11. [Putnam **2008 A3**] Start with a finite sequence $a_1, a_2, \cdots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\mathrm{lcm}[a_j, a_k]$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)