MATH 7230 Homework 2 - Fall 2018

Due Wednesday, Sep. 12 at 1:30

`https://www.math.lsu.edu/%7Emahlburg/teaching/2018F-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Ash A.B.C" means Problem C from Section A.B in the textbook.

In Problems 1 and 2 you will prove properties of local rings and maximal ideals in a slightly different manner than found in Proposition 1.2.8 and Problem 1.2.1 from the text.

1. (a) Suppose that $I \subset R$ is a proper ideal. Prove that if $a \in I$, then $1 + a \notin I$.
   (b) Now suppose that $M \subset R$ is a maximal ideal (and note that the first part applies, since a maximal ideal must be proper). Show that if for all $m \in M$ it is also true that $1 + m$ is a unit, then R is a local ring.
   (c) Prove the converse: If R is a local ring and $M \subset R$ is a maximal ideal, then $1 + m$ is a unit for all $m \in M$.
   (d) Give an example of a nonlocal ring R and a maximal ideal M such that the condition in parts (b) and (c) fails.
2. In this problem you will determine whether maximality is necessary in Problem 1 (b). Suppose that $I \subset R$ is an ideal such that $1 + a$ is a unit for all $a \in I$.
   (a) Show that I is not necessarily a maximal ideal. This is best accomplished by finding an example!
   (b) Is it still true that R must be a local ring?
3. (a) Ash 1.2.2.
   (b) The first part shows that $\mathbb{Z}/n\mathbb{Z}$ is a local ring if n is a prime power. Is this condition necessary?
4. In this problem you will prove a special case of Ash 1.2.3 (though the general case is not significantly different).
   (a) Suppose that K is a field and consider the ring of polynomials $K[x]$. Briefly explain/recall why this is a Euclidean domain (and hence a PID, so prime and maximal ideals are equivalent).
   (b) Prove that $(x) \subset K[x]$ is a prime ideal.
   (c) Now consider the ring of rational functions

$$R := \left\{ \frac{f(x)}{g(x)} \;\middle|\; f, g \in K[x],\ g(0) \neq 0 \right\}.$$

   Follow the example described in lecture (for the localization $\mathbb{Z}_{(p)}$), and use Lemmas 1.2.2 and 1.2.3 to prove that R is a local ring. What is the (unique) maximal ideal?

5. Show that the condition that I is prime is **necessary** in Proposition 1.2.4 of Ash by giving an example where $I \subsetneq h^{-1}\left(S^{-1}R\right)$. One possibility is to let $R = \mathbb{Z}, S = \mathbb{Z} \setminus (3)$, and $I = (6) \subseteq \mathbb{Z}$. Now determine $h^{-1}\left(S^{-1}R\right)$.

6. (a) Ash 2.1.3. For the trace of $\theta\sqrt{3}$, specify which field extension $E/\mathbb{Q}$ you are using to calculate $\mathrm{Tr}_{E/\mathbb{Q}}$.

   (b) Did it matter which E you used in part (a)? Why or why not?

7. Ash 2.1.4. You may read the solution in Ash, but do **not** use this argument. Instead, argue by contradiction: suppose that $\sqrt{3} \in \mathbb{Q}[\theta]$, so that

$$x = \sqrt{3} = a + b\theta + c\theta^2 + d\theta^3, \qquad a, b, c, d \in \mathbb{Q}.$$

   Now show that it is impossible for such a representation to satisfy $x^2 = 3$.