MATH 7230 Homework 3 - Fall 2018

Due Wednesday, Sep. 19 at 1:30

`https://www.math.lsu.edu/%7Emahlburg/teaching/2018F-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Ash A.B.C" means Problem C from Section A.B in the textbook.

In Problems 1 – 2 you will give an alternative proof that if A is a subring of E, then the algebraic integers with respect to A form a subring of E. If you need a reference, this argument is found in Milne Chapter 2.

1. A polynomial $f(x_1, \ldots, x_n) \in A[x_1, \ldots, x_n]$ is *symmetric* if $f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_1, \ldots, x_n)$ for any permutation π. The elementary symmetric polynomials are defined by

$$S_k(x_1, \ldots, x_n) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} \qquad (0 \le k \le n).$$

   (a) For example, $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$ is a symmetric polynomial. Write f as a polynomial in S_1, S_2 and S_3.

   (b) Now prove that any symmetric function can be written as a polynomial in the S_k. *Hint: Your calculations from part (a) should suggest a natural way to order monomials for an inductive argument....*

2. Now suppose that $x \in E$ is integral over A, with minimal polynomial $\min_{x,A}(X) = f(X) \in A[X]$.

   (a) If the minimal polynomial factors (possibly in some extension of E) as $f(X) = (X - x_1) \cdots (X - x_n)$, with $x_1 = x$, show that $S_k(x_1, \ldots, x_n) \in R$ for all k.

   (b) Let x and y be integral elements in E (with respect to A), with minimal polynomials $f(X) = (X - x_1) \cdots (X - x_n)$ and $g(Y) = (Y - y_1) \cdots (Y - y_m)$, respectively. Define

$$F(X) := \prod_{j=1}^{m} f(X - y_j),$$

   which has as roots all sums $x_i + y_j$, including $x + y$. Using Problem 1, show that $F(X) \in A[X]$. This implies that $x + y$ is integral.
   Similarly, define a monic polynomial $G(X)$ that has xy as a root, and show that $G(X) \in A[X]$, so that xy is also integral.

In Problems 3–4 you will give two proofs of the Van der Monde determinant formula, which states that

$$V(x_1, \cdots, x_n) := \det \begin{vmatrix} 1 & 1 & c\ldots & 1 \\ x_1 & x_2 & c\ldots & x_n \\ \vdots & \vdots & c\ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & c\ldots & x_n^{n-1} \end{vmatrix} = \prod_{i>j} (x_i - x_j). \qquad (1)$$

3. Give a proof by **induction.** You should start by using the row operations $R_j \mapsto R_j - x_1^{j-1} R_1$ for $2 \le j \le n$, which give

$$V(x_1, \cdots, x_n) = \det \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & \cdots & x_n - x_1 \\ 0 & x_2^2 - x_1^2 & \cdots & x_n^2 - x_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-1} - x_1^{n-1} & \cdots & x_n^{n-1} - x_1^{n-1} \end{vmatrix}.$$

Now apply the additional row operations (in order):

$$R_n \mapsto R_n - x_1 R_{n-1}, \quad R_{n-1} \mapsto R_{n-1} - x_1 R_{n-2}, \ \cdots, \ R_3 \mapsto R_3 - x_1 R_2.$$

If you've done this correctly, the induction step should now be clear.

4. Now prove the formula using **polynomials**, as if you view the matrix entries as variables, then you can use unique factorization in $R[X_1, \cdots, X_n]$ (where R is any UFD). In particular, show that both sides of (1) have the same roots, degree, and leading coefficient.

In Problems 5–6 you will prove Ash Theorem 2.2.9, which states that if $(x, y) : V \to F$ is a nondegenerate, symmetric, bilinear ($(ax + bw, y) = a(x, y) + b(w, y)$) form on V (an n-dimensional F-vector space), then any basis $V = \langle x_1, \cdots, x_n \rangle_F$ has a *dual basis* (referred to V) $V = \langle y_1, \cdots, y_n \rangle_F$ such that $(x_i, y_j) = \delta_{ij}$. If you are unfamiliar with the terms, *nondegenerate* means that for each $x \neq 0$, there is some y such that $(x, y) \neq 0$; *symmetric* means that $(x, y) = (x, y)$ for all x, y; and *bilinear* means that for $x, v, w, y \in V$ and $a, b \in F$, $(ax + bv, y) = a(x, y) + b(v, y)$ and $(x, ay + bw) = a(x, y) + b(x, w)$.

Note that the canonical orthogonal basis $\langle e_1, \cdots, e_n \rangle_F$, with $e_j := (0, \cdots, 0, 1, 0, \cdots 0)$, is its own dual.

5. (a) Ash 2.2.6. A *linear form* on V is a map $f : V \to F$ such that $f(ax + bv) = af(x) + bf(v)$. The *dual space* of V is

$$V^* := \{ f \mid f \text{ is a linear form on } V \}.$$

Note that V^* is also an F-vector space, and these problems essentially show that when $V = F^n$ is finite-dimensional, then $V^* \cong F^n$ as well.

Remark: In general, when V is infinite-dimensional V^* can have larger dimension!

(b) Ash 2.2.7. The "high-level" proof is to appeal to kernels/null spaces/dimension-counting for finite-dimensional vector spaces. However, given a linear form $f : V \to F$, it is also possible to construct y such that $f = \ell(y)$ using the canonical basis.

6. (a) Ash 2.2.8. If $\langle x_1, \cdots, x_n \rangle_F$ is a basis for V, then the corresponding *dual basis* is

$$\langle f_1, \cdots, f_n \in V^* \mid f_j(x_i) = \delta_{ij} \rangle_F.$$

(b) Ash 2.2.9.