**MATH 7230 Homework 4 - Fall 2018**

Due Wednesday, Oct. 3 at 1:30

`https://www.math.lsu.edu/%7Emahlburg/teaching/2018F-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Ash A.B.C" means Problem C from Section A.B in the textbook.

1. In this problem you will prove Stickelberger's Theorem, which states that if $L/\mathbb{Q}$ is a number field and $x_1, \cdots, x_n \in B \subset L$ are algebraic integers, then the discriminant satisfies
$$D(x_1, \cdots, x_n) \equiv 0, 1 \pmod 4.$$
This is essentially the proof outlined in Ash Problems 2.3.1 – 2.3.3.
   (a) First, recall from Lemma 2.3.3 that
$$D(x_1, \cdots, x_n) = \left(\det\left(\sigma_i(x_j)\right)_{1 \leq i,j \leq n}\right)^2, \tag{1}$$
where the $\sigma_i$ are the distinct $\mathbb{Q}$-embeddings of $L$ (into an algebraic closure). Using the definition of the determinant, write
$$\det\left(\sigma_i(x_j)\right)_{1 \leq i,j \leq n} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} \sigma_i\left(x_{\pi(i)}\right) =: P - N,$$
where $P$ is the sum over even permutations $\pi \in S_n$, and $N$ is the sum over odd $\pi$.
   (b) Now suppose $\sigma$ is an $\mathbb{Q}$-embedding, and consider its action on $P$ and $N$. First, explain why there is a permutation $\tau \in S_n$ such that $\sigma(\sigma_i)) = \sigma_{\tau(i)}$ for all $i$. Then show that
$$\sigma(P) = \sum_{\substack{\pi \in S_n \\ \operatorname{sgn}(\pi) = \operatorname{sgn}(\tau)}} \prod_{i=1}^{n} \sigma_i\left(x_{\pi(i)}\right) = \begin{cases} P & \text{if } \operatorname{sgn}(\tau) = 1; \\ N & \text{if } \operatorname{sgn}(\tau) = -1. \end{cases}$$
   (c) Use part (b) to conclude that $P + N$ and $PN$ are invariant under all $\sigma$, and are thus in $\mathbb{Q}$. Now use the fact that the $x_j$ are algebraic integers to conclude that $P + N, PN \in \mathbb{Z}$.
   (d) Finally, use $D = (P - N)^2$ to conclude the theorem statement.
2. Find $B$, the set of algebraic integers (over $\mathbb{Z}$) in the field $L = \mathbb{Q}[x]/\left(x^3 - 2x + 5\right)$.

   *Hint: Use Ash Problems 2.3.4 and 2.3.5, as discussed in lecture.*
3. Prove the basic fact about prime ideals mentioned in Ash 3.2.1: If $I_1, \cdots, I_n \subset R$ are ideals, and $I_1 \cdots I_n \subset P$, where $P \subset R$ is a prime ideal, then $I_j \subset P$ for some $j$.

*Hint: Suppose that $I_1, \cdots, I_{n-1} \subsetneq P$, so there are elements $a_j \in I_1 \setminus P$ for $j = 1, \cdots, n-1$. Now consider $a_1 a_2 \cdots a_{n-1} a$ for an arbitrary element $a \in I_n$.*

*Remark: Writing $A \mid B$ for (reverse) ideal containment, i.e. $B \subset A$, this fact becomes $P \mid I_1 \cdots I_n \implies P \mid I_j$ for some $j$.*

4. In this problem you will prove some of the basic properties of Noetherian rings. Recall that $R$ is Noetherian if it satisfies the ascending chain condition (ACC), which means that any chain of ideals $I_j \subset R$ eventually stabilizes:

$$I_1 \subset I_2 \subset I_3 \subset \cdots \quad \implies \quad \text{There is some } n \text{ such that } I_m = I_n \text{ for } m \geq n.$$

   (a) Prove that $R$ is Noetherian if and only if every non-empty set $S$ of ideals in $R$ contains a maximal element $J \in S$ (in the sense that $J \subset I$ and $I \in S$ implies that $I = J$ or $R$).

   *Hint: For the forward direction, it is convenient to argue by contradiction, so let $S$ be a set of ideals that does* **not** *contain a maximal element. Now obtain a contradiction to ACC.*

   (b) Prove that $R$ is Noetherian if and only if every ideal $I \subset R$ is finitely generated.

   *Hint: For the forward direction, fix an ideal $I$, and let $S$ be the set of all ideals generated by a finite collection of elements in $I$. Now use part (a)....*

5. This problem gives an alternative proof (using induction) of the Prime Avoidance Lemma, which is found in Ash Problems 3.1.1 – 3.1.3. This will be needed for Ash Theorem 8.1.2. The statement is as follows:

   Suppose that $I_1, I_2, \cdots, I_s \subset R$ are ideals, with $I_3, \cdots, I_s$ all prime. If $J \subset R$ is another ideal and $J \not\subset I_i$ for any $i$, then $J \not\subset (I_1 \cup \cdots \cup I_s)$.

   (a) First, argue that without loss of generality one may assume that $I_i \not\subset I_j$ for any $i, j$.

   (b) Next, prove the case $s = 2$. For $i = 1, 2$, let $a_1 \in J \setminus I_i$, and show that $a_1 + a_2 \notin I_1 \cup I_2$.

   (c) Now induct on $s$. In particular, if $J \not\subset I_i$ for $1 \leq i \leq s-1$, then the inductive step gives some $a \in J \setminus (I_1 \cup \cdots \cup I_{s-1})$. If $a \notin I_s$, the proof is finished (why?), so assume $a \in I_s$.

   Use the fact that $I_s$ is prime to show that $JI_1 \cdots I_{s-1} \not\subset I_s$. Finally, choose $b \in JI_1 \cdots I_{s-1} \setminus I_s$, and consider $a + b$.

In Problems 6 – 8 you will explore ideal arithmetic in $R = \mathbb{Z}[\sqrt{-5}]$.

6. Note that these are all quite short, and it is very likely that you have already performed some of these calculations (this is a common first example of an arithmetic ring that is not a UFD). Feel free to skip parts that you are already familiar with.

   (a) Ash 3.4.1.

   (b) Ash 3.4.2.

   (c) Ash 3.4.3.

(d) Ash 3.4.4.

(e) Ash 3.4.5.

7. We will soon learn that the class group of $R$ has order 2, which has the amazing consequence that the product of **any** two non-principal fractional ideals must be principal!

   (a) As in Ash, let $P_2 = (2, 1+\sqrt{-5})$. Verify that
   $$P = \left\{a + b\sqrt{-5} \mid a \equiv b \pmod 2\right\}.$$

   (b) Write $P_3 = (3, 1+\sqrt{-5})$ in a similar way.

   (c) Determine the principal ideal produced by the product of the two ideals above. In other words, find $\alpha \in R$ such that $P_2P_3 = (\alpha)$.

8. Finally, determine the prime factorization of the following fractional ideals in $R$. You will have an easier time recognizing the prime ideals if you have completed Problems 6 and 7.

   (a) $I = \left\langle \frac{3}{5} \right\rangle_R$,

   (b) $I = \left\langle \frac{1}{2}, \frac{1}{1+\sqrt{-5}} \right\rangle_R$.