MATH 7230 Homework 6 - Fall 2018

Due Wednesday, Oct. 24 at 1:30

`https://www.math.lsu.edu/~mahlburg/teaching/2018F-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Ash A.B.C" means Problem C from Section A.B in the textbook.

1. (a) Ash 5.3.1.
   (b) Ash 5.3.2.
2. (a) Ash 5.3.3.
   (b) Ash 5.3.4.
3. (a) Ash 5.3.7.
   (b) Ash 5.3.8.
4. (a) Ash 5.3.9.
   (b) Ash 5.3.10.

In Problems 5 – 6, you will consider cubic number fields L, i.e. $n = [L : \mathbb{Q}] = 3$, with discriminant $d = \mathrm{Disc}_{L/\mathbb{Q}}$.

5. (a) Use Minkowski's Bound to give an absolute lower bound for $|d|$.

   Remark: The cubic number field with the smallest discriminant (in magnitude) is actually $\mathbb{Q}[X]/(X^3 - X + 1)$*, which has* $d = -23$.

   (b) Suppose that L has $r_2 = 1$ (so that it has complex embeddings). Show that $h_L = 1$ if $|d| \leq 49$.

6. Let $L = \mathbb{Q}[X]/(f(X))$ with $f(X) := X^3 - X^2 - 2X + 1$. This is a "totally real" cubic field, which means that there are no complex embeddings (i.e., $r_1 = 3, r_2 = 0$). This is due to the fact that $f(X)$ has three real roots (which can be shown using Descartes' Rule of Signs).

   (a) This example is often given instead as $L = \mathbb{Q}[X]/(g(X))$, with $g(x) := X^3 + X^2 - 2X - 1$ (for example, at `https://en.wikipedia.org/wiki/Cubic_field`. Explain why this is the same field.

   (b) Explain why the polynomial discriminant is invariant under linear shifts, i.e. $\mathrm{Disc} f(X + a) = \mathrm{Disc} f(X)$.

   (c) It is a fact that $B = \mathcal{O}_L$ has a power basis, so $B = \mathbb{Z}[\theta] \cong \mathbb{Z}[X]/(f(X))$. Show that $d = 49$.

   Hint: One approach is to use part (a) and Proposition 2.3.5, along with $\mathrm{Disc}\left(X^3 + aX + b\right) = -4a^3 - 27b^2$.

   (d) Use Minkowski's Bound to determine the class group of L.

   Remark: L has the smallest positive discriminant among all cubic number fields.