MATH 7230 Homework 9 - Fall 2018

Due Wednesday, Nov. 28 at 1:30

https://www.math.lsu.edu/~mahlburg/teaching/2018F-MATH7230.html

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

The notation "Ash A.B.C" means Problem C from Section A.B in the textbook.

Problems

1. In lecture we discussed the construction of the field completion of K with respect to the absolute value $|\bullet|$, which is defined to be the (ring) quotient of Cauchy sequences in K relative to the maximal ideal of null sequences:

$$\widehat{K}_{|\bullet|} := C_{K,|\bullet|}/N_{K,|\bullet|}.$$

The coset representatives are Cauchy sequences $c := \{c_n\}_n$ in K, and the absolute value is extended to $\widehat{K}_{|\bullet|}$ by setting

$$|c| := \lim_{n\to\infty} |c_n|.$$

We sketched the proof that $\widehat{K}_{|\bullet|}$ is indeed a complete field, but left some details unresolved.

Suppose that $\alpha = \{\alpha_n\}_n$ is a Cauchy sequence in $\widehat{K}_{|\bullet|}$, so that each element has a representative $\alpha_n \equiv \{c_{n,m}\}_m \pmod{N_{K,\bullet}}$ that is a Cauchy sequence in K. This means that for any $\varepsilon > 0$ there is an $M(n,\varepsilon)$ such that

$$\left|c_{n,m} - c_{n,m'}\right| < \varepsilon \qquad \text{for } m, m' \geq M(n,\varepsilon).$$

Furthermore, the fact that α is a Cauchy sequence means that for any $\delta > 0$, there is an $N(\delta)$ such that

$$|\alpha_n - \alpha_{n'}| = \lim_{m\to\infty} \left|c_{n,m} - c_{n',m}\right| < \delta \qquad \text{for } n, n' \geq N(\delta).$$

Set

$$M(n) := M\left(n, \frac{1}{n}\right),$$

and define the sequence $d = \{d_n\} := \{c_{n,M(n)}\}$. The claim is that d is a Cauchy sequence (and thus an element of $\widehat{K}$), and that α converges to d.

(a) First, show that for any constant d and Cauchy sequence $c = \{c_n\}_n$, the expression $|c - d|$ is well-defined; i.e., that

$$\lim_{n\to\infty} |c_n - d| \qquad \text{exists.}$$

(b) Now show that if $m := \lceil \frac{3}{\varepsilon} \rceil$ and $n, n' \geq \max\left\{m, N\left(\frac{1}{m}\right)\right\}$, then

$$|d_n - d_{n'}| < \varepsilon.$$

Hint: Use the triangle inequality and compare to α_n and $\alpha_{n'}$ – as the choice of m suggests, you should split into three terms!

(c) Finally, prove that $\alpha \to d$, i.e., that $\lim_{n\to\infty} |\alpha_n - d_n| = 0$.

Remark:

2. (a) Ash 9.4.7.

   (b) Ash 9.4.8.

   (c) Ash 9.4.9.

3. In this problem you will prove some strange facts about the topology of $\mathbb{Q}_p$. In general, an absolute value $|\bullet|$ on K induces a metric by setting

$$d_{K,|\bullet|}(x, y) := |x - y|.$$

For the rest of the problem assume that $|\bullet|$ is a nontrivial non-archimedean absolute value on K, with metric d. The corresponding topology is given by the (open) balls

$$B_{K,|\bullet|}(x, r) := \{y \in K \mid d(x, y) < r\}.$$

   (a) Suppose that $x \neq y \in K$. Show that if $r \leq d(x, y)$, then $B(x, r) \cap B(y, r) = \emptyset$.

   (b) Use part (a) to show that if $y \notin B(x, r)$, then there is an open set $B(y, s)$ such that $B(y, s) \cap B(x, r) = \emptyset$. Conclude that $B(x, r)$ is closed!

   (c) Show that $K_{|\bullet|}$ is *totally disconnected*, which means that the only connected subsets are singletons. In other words, for any $S \subset K$ with two or more elements, there exist disjoint open sets U and V such that $S = U \cup V$.

4. (a) Prove that $\mathbb{Q}_p$ is uncountable.

   (b) Is $\mathbb{Z}_p$ also uncountable?

   Hint: Recall that Cantor's diagonalization argument for the reals is typically presented in such a way that shows that $(0, 1)$ is uncountable....