When n_1 and n_2 are relatively prime, the divisors of $n_1 n_2$ are the products of the divisors of n_1 and n_2, hence the sum we have obtained is a multiplicative function of n. When n is a prime power, say $n = p^r$, we use $\phi(p^j) = p^j - p^{j-1}$ for $j \geq 1$ to evaluate the sum as

$$\sum_{d|n} \frac{\phi(d)n^2}{d^2} = p^{2r} + \sum_{j=1}^{r} (p^{2r-j} - p^{2r-j-1}) = p^{2r} + p^{2r-1} - p^{r-1}.$$

The result follows.

Also solved by R. Bittencourt (Brazil), R. Brase, R. Chapman (U. K.), K. Gatesman, Y. J. Ionin, P. Lalonde (Canada), O. P. Lossers (Netherlands), M. A. Prasad (India), I. Sfikas, N. C. Singer, A. Stadler (Switzerland), M. Tang, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Divergence of a Series

12004 [2017, 755]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $a_1, a_2, \ldots$ be a strictly increasing sequence of real numbers satisfying $a_n \leq n^2 \ln n$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} 1/(a_{n+1} - a_n)$ diverges.

Solution by Nicholas C. Singer, Annandale, VA. For $k \geq 1$, apply the Harmonic-Mean–Arithmetic-Mean inequality to the positive numbers in $\{a_{2^k+j} - a_{2^k+j-1} \colon 1 \leq j \leq 2^k\}$ to obtain

$$\frac{1}{a_{2^k+1} - a_{2^k}} + \frac{1}{a_{2^k+2} - a_{2^k+1}} + \cdots + \frac{1}{a_{2^{k+1}} - a_{2^{k+1}-1}} \geq \frac{4^k}{a_{2^{k+1}} - a_{2^k}} \geq \frac{4^k}{a_{2^{k+1}} - a_1}.$$

Since $a_1 \leq 0$,

$$\frac{4^k}{a_{2^{k+1}} - a_1} = \frac{4^k}{a_{2^{k+1}} + |a_1|} \geq \frac{4^k}{2^{2k+2}(k+1)\ln 2 + |a_1|} = \frac{1}{4(k+1)\ln 2 + |a_1|/4^k}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{a_{n+1} - a_n} = \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \frac{1}{a_{2^k+j} - a_{2^k+j-1}} \geq \sum_{k=0}^{\infty} \frac{1}{4(k+1)\ln 2 + |a_1|/4^k} = \infty.$$

Editorial comment. Several solvers overlooked the possibility that a_n might be negative for some (or all) n.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Boukharfane (France), P. Bracken, R. Brase, H. Chen, P. J. Fitzsimmons, D. Fleischman, E. J. Ionaşcu, M. Javaheri, P. Komjáth (Hungary), O. Kouba (Syria), K. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), V. Mikayelyan (Armenia), P. Perfetti (Italy), Á. Plaza & K. Sadarangani (Spain), M. A. Prasad (India), J. C. Smith, O. Sonebi (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), GCHQ Problem Solving Group (U. K.), and the proposer.

A Suspicious Formula Involving Pi

12006 [2017, 970]. *Proposed by Jonathan D. Lee, Merton College, Oxford, U. K., and Stan Wagon, Macalester College, St. Paul, MN.* When n is an integer and $n \geq 2$, let $a_n = \lceil n/\pi \rceil$ and $b_n = \lceil \csc(\pi/n) \rceil$. The sequences $a_2, a_3, \ldots$ and $b_2, b_3, \ldots$ are, respectively,

$$1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \ldots$$

and

$$1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \ldots.$$

They differ when $n = 3$. Are they equal for all larger n?

Solution by Albert Stadler, Herrliberg, Switzerland. The answer is no, as can be checked by direct calculation for $n = 80143857$. As motivation for this answer, the Laurent expansion of $\csc(\pi x)$ is $1/(\pi x) + \pi x/6 + \cdots$ with all coefficients positive. Thus when $n \geq 2$ we have $0 < \csc(\pi/n) - n/\pi \leq \csc(\pi/2) - 2/\pi < 1$. It follows that $b_n - 1 \leq a_n \leq b_n$, and furthermore that $b_n = a_n + 1$ when there exists an integer m such that

$$0 < \frac{m}{n} - \frac{1}{\pi} < \frac{\pi}{6n^2}. \qquad (*)$$

Good candidates for m/n are given by the continued fraction convergents of $1/\pi$, every second one of which is greater than $1/\pi$. The continued fraction representation of $1/\pi$ is $[0; 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots]$, and so one may compute that the first two convergents that satisfy $(*)$ are the second and 14th. These are $1/3$ and $25510582/80143857$, leading to $a_n \neq b_n$ for $n = 3$ and $n = 80143857$.

Editorial comment. Direct computation shows that $a_n = b_n$ when $4 \leq n \leq 80143856$.

It is natural to wonder whether the sequences differ infinitely often. The proposers noted that by Hurwitz's theorem there are infinitely many convergents to $1/\pi$ such that $|\frac{1}{\pi} - \frac{m}{n}| < \frac{1}{\sqrt{5}n^2}$, which implies $|\frac{1}{\pi} - \frac{m}{n}| < \frac{\pi}{6n^2}$. However, only even-numbered convergents will be greater than $1/\pi$, as needed for $(*)$. It seems likely, given how the continued fraction of π is expected to behave, that there are infinitely many even-numbered convergents among the ones that satisfy the condition of Hurwitz's theorem, but this is currently unresolved.

Also solved by A. Berele, R. Chapman (U. K.), S. Demers (Canada), G. Fera (Italy), O. P. Lossers (Netherlands), M. D. Meyerson, V. Mikayelyan (Armenia), M. Reid, C. Schacht, V. Schindler (Germany), J. C. Smith, A. Stenger, A. Stewart, R. Stong, W. Stromquist, R. Tauraso (Italy), D. Terr, H. Widmer (Switzerland), L. Zhou, Armstrong Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposers.

An Application of the Phragmén–Lindelöf Principle

12009 [2017, 970]. *Proposed by George Stoica, Saint John, NB, Canada.* Find all continuous functions $f : [0, 1] \to \mathbb{R}$ satisfying $\left|\int_0^1 e^{xy} f(x)\, dx\right| < 1/y$ for all positive real numbers y.

Solution by James Christopher Smith, Knoxville, TN. We claim that the only such function is the constant 0. Let $g(z) = \int_0^1 e^{xz} f(x)\, dx$ for all $z \in \mathbb{C}$. Because f is continuous on $[0, 1]$, it is bounded and measurable, so g is an entire function.

We apply the Phragmén–Lindelöf principle to $g(z)$ on the first quadrant D in the complex plane. First, we note the estimate

$$|g(z)| \leq \int_0^1 \left|e^{xz} f(x)\right| dx \leq Me^{|z|},$$

where $M = \int_0^1 |f(x)|\, dx$. Second, we claim that g is bounded on the real axis. Indeed, when $-\infty < y \leq 1$ we have $|g(y)| \leq Me$ and for $y \geq 1$ we have $|g(y)| \leq 1/y \leq 1$. And third, we claim that g is bounded on the imaginary axis. Indeed, for $y \in \mathbb{R}$ we have $|g(iy)| \leq \int_0^1 \left|e^{ixy} f(x)\right| dx \leq M$. Therefore, by the Phragmén–Lindelöf principle, $g(z)$ is bounded in the quadrant D. Similarly, $g(z)$ is bounded in each of the other three quadrants as well.

Thus $g(z)$ is a bounded entire function, so by Liouville's theorem $g(z)$ is constant. Hence, for all $n \geq 1$, we have $0 = g^{(n)}(0) = \int_0^1 x^n f(x)\, dx$. By the Weierstrass approximation theorem applied to $xf(x)$, we conclude that f is the constant function 0.

Also solved by K. F. Andersen (Canada), A. Stadler (Switzerland), G. Vidiani (France), GCHQ Problem Solving Group (U. K.), and the proposer.