Since $11661/30 < 400$ and $3549/30 < 200$,

$$400e^{20t} + 200e^{-10t} > \frac{11661}{30}e^{20t} + \frac{3549}{30}e^{-10t} \geq 507e^{13t},$$

which is $(*)$.

It follows by Jensen's inequality that $f(u) + f(v) + f(w) \geq 3f(0) = 0$ with equality only if $u = v = w = 0$.

Editorial comment. Several solvers noted the similarity between this problem and problem 11543 [2010, 390; 2012, 609], which asked for a proof of the inequality $(x^5 + y^5 + z^5)^2 \geq 3(x^7 + y^7 + z^7)$, where x, y, and z are positive numbers with $xyz = 1$. Replacing x^{10}, y^{10}, and z^{10} in the present problem with u, v, and w, the required inequality becomes $(u + v + w)^2 \geq 3(u^{1.3} + v^{1.3} + w^{1.3})$, subject to the constraints $u, v, w > 0$ and $uvw = 1$. Similarly, replacing x^5, y^5, and z^5 in problem 11543 with u, v, and w leads to the inequality $(u + v + w)^2 \geq 3(u^{1.4} + v^{1.4} + w^{1.4})$, with the same constraints. This led some solvers to consider the generalization $(u + v + w)^2 \geq 3(u^a + v^a + w^a)$. For fixed positive u, v, and w satisfying $uvw = 1$, the quantity $u^a + v^a + w^a$ is an increasing function of $a \geq 0$. This shows that the inequality in problem 11543 implies the inequality in the present problem, thus providing an alternative solution. Michael Reid and (independently) the GCHQ Problem Solving Group investigated the largest value of a for which the generalized inequality is valid, and they found numerically that it is approximately 1.4047557.

Also solved by P. Bracken, P. P. Dályay (Hungary), G. Fera (Italy), K. Gatesman, L. Giugiuc (Romania), D. Glazkov (Russia), M. Kauers & D. Zeilberger, K. T. L. Koo (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Loverde, H. L. Nhat (Vietnam), C. Pranesachar (India), M. Reid, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, AN-anduud Problem Solving Group (Mongolia), GCHQ Problem Solving Group (UK), and the proposer.

A Chebyshev Determinant

12025 [2018, 180]. *Proposed by Askar Dzhumadil'daev, S. Demirel University, Almaty, Kazakhstan.* The Chebyshev polynomials of the second kind are defined by the recurrence relation $U_0(x) = 1$, $U_1(x) = 2x$, and $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ for $n \geq 2$. For an integer n with $n \geq 2$, prove

$$\det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ x & 0 & 1 & \cdots & 1 & 1 \\ x^2 & x & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & x^{n-4} & \cdots & 0 & 1 \\ x^{n-1} & x^{n-2} & x^{n-3} & \cdots & x & 0 \end{bmatrix} = (-1)^{n-1}x^{n/2}U_{n-2}(\sqrt{x}).$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA. Let A_n denote the given $n \times n$ matrix, and let B_n denote the matrix obtained from A_n by changing its (n, n) entry to 1. Thus

$$\det B_n = \det A_n + \det A_{n-1}.$$

We use this formula to find a recurrence relation for $\det A_n$. Subtract column n from column $n-1$ in A_n and denote the new matrix A'_n. Evaluate $\det A'_n$ by expanding along column $n-1$. When column $n-1$ and row $n-1$ are deleted from A'_n and x is factored out of its last row, the resulting matrix is A_{n-1}. When column $n-1$ and row n are deleted from A'_n, the resulting matrix is B_{n-1}. Therefore

$$\det A_n = \det A'_n = -x \det A_{n-1} - x \det B_{n-1} = -2x \det A_{n-1} - x \det A_{n-2}.$$

By inspection, the values of $\det A_n$ for $n \in \{2, 3\}$ are as claimed. We confirm the remaining values by induction. For $n \geq 4$,

$$\det A_n = -2x(-1)^n x^{(n-1)/2} U_{n-3}(\sqrt{x}) - x(-1)^{n-1} x^{(n-2)/2} U_{n-4}(\sqrt{x}).$$

We claim that the right side equals $(-1)^{n-1} x^{n/2} U_{n-2}(\sqrt{x})$. Dividing the claimed equation by $(-1)^{n-1} x^{n/2}$ yields the equivalent equation

$$2\sqrt{x} U_{n-3}(\sqrt{x}) - U_{n-4}(\sqrt{x}) = U_{n-2}(\sqrt{x}),$$

which is the recurrence for $U_{n-2}(\sqrt{x})$. This completes the proof.

Also solved by U. Abel and V. Kushnirevych (Germany), K. F. Andersen (Canada), R. Chapman (UK), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), N. Grivaux (France), W. Johnson, B. Karaivanov (USA) & T. S. Vassilev (Canada), D. Kim (South Korea), K. T. L. Koo (China), O. Kouba (Syria), B. Kurmanbek (Kazakhstan), P. Lalonde (Canada), M. Lerma, O. P. Lossers (Netherlands), M. Naidu, M. Omarjee (France), A. Pathak, M. Reid, C. Sanford, J. H. Smith, A. Stadler (Switzerland), R. Stong, J. Stuart, J. Van hamme (Belgium), J. Vinuesa (Spain), T. Wiandt, L. Zhou, AN-anduud Problem Solving Group (Mongolia), GCHQ Problem Solving Group (UK), and the proposer.

A Series with Harmonic Numbers

12026 [2018, 180]. *Proposed by Michel Bataille, Rouen, France.* For $n \in \mathbb{N}$, let $H_n = \sum_{k=1}^n 1/k$ and $S_n = \sum_{k=1}^n (-1)^{n-k}(H_1 + \cdots + H_k)/k$. Find $\lim_{n\to\infty} S_n/\ln n$ and $\lim_{n\to\infty} (S_{2n} - S_{2n-1})$.

Solution by Douglas B. Tyler, Torrance, CA. The limits are $1/2$ and $1 + (\ln(2))^2 - \ln(2) - \pi^2/6$, respectively.

Note first that

$$\sum_{j=1}^k H_j = \sum_{j=1}^k \sum_{i=1}^j \frac{1}{i} = \sum_{i=1}^k \sum_{j=i}^k \frac{1}{i} = \sum_{i=1}^k \frac{k+1-i}{i} = (k+1)H_k - k.$$

Thus

$$S_{2n} = \sum_{k=1}^{2n} (-1)^k \left(\left(1 + \frac{1}{k}\right) H_k - 1 \right) = \sum_{k=1}^{2n} (-1)^k H_k - \sum_{k=1}^{2n} (-1)^{k-1} \frac{H_k}{k}.$$

The first sum on the right side is $1/2 + \cdots + 1/(2n) = H_n/2 = (\ln n + \gamma)/2 + o(1)$. The second sum is alternating and converges to $A = \sum_{k=1}^\infty (-1)^{k-1} H_k/k$. Thus $S_{2n} = (\ln n + \gamma)/2 - A + o(1)$.

Now the terms of S_{2n-1} are opposite to those of S_{2n}, as far as they go, so

$$S_{2n-1} = -S_{2n} + \frac{1}{2n} \sum_{k=1}^{2n} H_k = -S_{2n} + \left(1 + \frac{1}{2n}\right) H_{2n} - 1.$$

Now $S_{2n} = (\ln n)/2 + O(1)$ and $S_{2n-1} = \ln n/2 + O(1)$ yield $\lim_{n\to\infty} S_n/\ln n = 1/2$.

To compute the second limit, note that $S_{2n} - S_{2n-1} = 2S_{2n} - H_{2n} + 1 + o(1) = 1 - \ln 2 - 2A + o(1)$. To evaluate A, let $f(x) = \sum_{k=1}^\infty (-1)^{k-1} x^k H_k/k$ for $|x| < 1$. Now f is analytic, and $A = f(1)$ by Abel's limit theorem. Also

$$f'(x) = \sum_{k=1}^\infty (-x)^{k-1} H_k = \sum_{k=1}^\infty \sum_{j=1}^k \frac{(-x)^{k-1}}{j}$$

$$= \sum_{j=1}^\infty \sum_{k=j}^\infty \frac{(-x)^{k-1}}{j} = \sum_{j=1}^\infty \frac{(-x)^{j-1}}{j(1+x)} = \frac{\ln(1+x)}{x(1+x)}.$$