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1 Problem Statement

When n is an integer and $n \geq 2$, let $a_n = \lceil \frac{n}{\pi} \rceil$ and $b_n = \lceil \csc \frac{\pi}{n} \rceil$. The sequences $a_2, a_3, \dots$ and $b_2, b_3, \dots$ are, respectively,

$$1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \dots$$

and

$$1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \dots$$

They differ when $n = 3$. Are they equal for all larger n?

The answer is **No,** as there is at least one integer value for which the ceiling functions of the respective functions differ for said integer value. Using a high-precision calculator, it can be shown that for $n = 80143857$,

$$a_n = \frac{80143857}{\pi} = 25510581.999999995\dots$$
$$b_n = \csc\left(\frac{\pi}{80143857}\right) = 25510582.000\dots$$

Thus proving that a_n and b_n are **not** equal for all positive integer values of n.

However, the value $n = 80143857$ was not obtained by "brute force", and the remainder of this solution is dedicated to revealing how this value was extrapolated from examining extreme cases where the two functions are "close" to an integer threshold.

2 Details of Solution

2.1 Limiting behavior

We can observe the functions in general for all $x > 0$. It is worth examining the ratio of these functions as $x \to \infty$ so as to compare their respective behaviors as x becomes "large". We assign the following notation to each function

$$f(x) := \frac{x}{\pi},$$
$$g(x) := \csc\frac{\pi}{x}.$$

So, to examine the limiting behavior,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x/\pi}{1/\sin(\pi/x)} = \lim_{x\to\infty} \frac{\sin(\pi/x)}{\pi/x}$$

Now it is apparent that both the numerator and denominator converge to an indeterminate as x approaches infinity. We can therefore employ the L'Hospital's rule, obtaining

$$\lim_{x\to\infty} \frac{\sin(\pi/x)}{\pi/x} = \lim_{x\to\infty} \frac{\cos(\pi/x) \times -\pi/x^2}{-\pi/x^2} = \lim_{x\to\infty} \cos(\pi/x) = 1$$

Thus it has been shown that the functions $f(x)$ and $g(x)$ approach one another as x approaches infinity. This means that it is possible that the respective ceiling functions of the functions will always be equal due to the functions being so similar. However, as we will see, it is important to examine extreme cases where both functions are "close" to an integer value.

2.2 Using Taylor Series to establish which function is greater

We now introduce the Taylor Series of the sine function so as to aid in evaluating the ratio of the functions,

$$\sin\left(\frac{\pi}{x}\right) = \frac{\pi}{x} - \frac{(\frac{\pi}{x})^3}{3!} + \frac{(\frac{\pi}{x})^5}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{(\pi/x)^{(2n+1)!}}{(2n+1)!}.$$

Therefore, we can rewrite the ratio of functions as

$$\frac{f(x)}{g(x)} = \frac{x/\pi}{1/\sin(\pi/x)} = \frac{\sin(\pi/x)}{\pi/x} = \frac{\frac{\pi}{x} - \frac{(\frac{\pi}{x})^3}{3!} + \frac{(\frac{\pi}{x})^5}{5!} - \cdots}{\frac{\pi}{x}} = 1 - \frac{(\frac{\pi}{x})^2}{3!} + \frac{(\frac{\pi}{x})^4}{5!} - \cdots.$$

For $x > \pi$ the Alternating Series test implies that the series is bounded as such,

$$1 - \frac{(\frac{\pi}{x})^2}{3!} < \frac{f(x)}{g(x)} < 1. \tag{1}$$

Therefore, using this information we obtain

$$\frac{f(x)}{g(x)} < 1 \implies g(x) > f(x)$$

This establishes $f(x)$ as the smaller function and is therefore useful in determining whether both functions are above or below an integer threshold.

2.3 Examining Ceilings

In order for the ceiling functions to differ, there must exist an integer value, hereby denoted as m, such that

$$\frac{x}{\pi} < m \leq \csc\left(\frac{\pi}{x}\right)$$

This is most likely to happen in cases where $\frac{x}{\pi}$, for some positive integer value of x, is "close" to the integer threshold m. Then

$$\frac{x}{\pi} \approx m \iff \pi \approx \frac{x}{m}$$

It is clear then that integer ratios that closely approximate π should first and foremost be examined for a proof by counterexample. It is then worth considering the continued fraction expansion of π so as to extrapolate integer ratios corresponding to π, as it is known that the finite *convergents* (integer ratios that occur if you stop at any point along the continued fraction) give very good rational approximations. The continued fraction expansion is (A001203 in [2])

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}\cdots}}}.$$

The convergents are (A002485 and A002486 in [2])

$$\left\{3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \ldots, \frac{80143857}{25510582}, \ldots\right\}. \tag{2}$$

The reader will recognize the numerator of the last fraction $\frac{80143857}{25510582}$ as the integer value that produced differing a_n and b_n values. This was the thirteenth element of the series of π convergents.

3 Further Discussion

It is worth noting that the solutions to this problem need not be a convergent of the continued fraction, and the author does not know if $n = 80143857$ is the minimal value such that $a_n \neq b_n$. If we are looking for an example in which the ceiling functions of the respective functions differ, then we are looking for the following criteria to be met

$$f(n) < m \leq g(n) \implies m - f(n) < g(n) - f(n).$$

Written explicitly,

$$m - \frac{n}{\pi} < \frac{1}{\sin \frac{\pi}{n}} - \frac{n}{\pi}.$$

Recalling the lower bound from (1),

$$\frac{1}{\sin \frac{\pi}{n}} - \frac{n}{\pi} < \frac{1}{\frac{\pi}{n} - \frac{(\frac{\pi}{n})^3}{3!}} - \frac{n}{\pi}$$

By some manipulation we obtain

$$\frac{1}{\frac{\pi}{n} - \frac{(\frac{\pi}{n})^3}{3!}} - \frac{n}{\pi} = \frac{\frac{n}{\pi}}{1 - \frac{(\frac{\pi}{n})^2}{3!}} - \frac{\frac{n}{\pi}(1 - \frac{(\frac{\pi}{n})^2}{3!})}{1 - \frac{(\frac{\pi}{n})^2}{3!}} = \frac{\frac{\pi}{6n}}{1 - \frac{\pi^2}{6n^2}},$$

Which, for very large values of n, closely approximates $\frac{\pi}{6n}$.

Recall that we are considering an instance where $\lceil \frac{n}{\pi} \rceil = m$, so

$$\frac{n}{\pi} \approx m.$$

Thus

$$m - \frac{n}{\pi} < (1+\varepsilon)\frac{\pi}{6n} \implies \pi - \frac{n}{m} < (1+\varepsilon)\frac{\pi^2}{6mn},$$

where $\varepsilon \to 0$ for large n. We can therefore write

$$\pi - \frac{n}{m} < (1+\varepsilon)\frac{\pi^2}{6mn} = (1+\varepsilon')\frac{\pi^2}{6m \cdot m\pi} = (1+\varepsilon')\frac{\pi}{6m^2},$$

where similarly, $\varepsilon' \to 0$. It is now worth introducing the following fact from Number Theory.

Theorem 1 ([1]) *If α is irrational, and*

$$\left|\alpha - \frac{n}{m}\right| < \frac{1}{2m^2},$$

then $\frac{n}{m}$ is a continued fraction convergent of α.

However, since $\frac{\pi}{6} = 0.523598\cdots > \frac{1}{2}$, Theorem 1 does not apply, and it is therefore not mandatory that an integer solution to this problem be a continued fraction convergent. In other words, there may be integer solutions less than the one proposed in this proof that are not elements of the series of π convergents.

As a final remark, it is an interesting question whether or not there are infinitely many n such that $a_n < b_n$. It can be checked that there are at least multiple such n. The numerator of the 15th convergent of π from (2) is $n = 245850922$, and for this value it is also true that $a_n < b_n$.

References

[1] I. Niven, H. Montgomery, and H. Zuckerman, *An Introduction to the Theory of Numbers, 5th Edition.* John Wiley & Sons, Inc., New York, 1991.

[2] The On-Line Encyclopedia of Integer Sequences, `oeis.org`. Accessed Apr. 25, 2018.