Chebyshev polynomials of the Second Kind and determinants of special matrices: American Math. Monthly Problem 12025

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1 Introduction

In this paper I will outline a solution to a problem from the Problems and Solutions section of The American Mathematical Monthly, **125:2**, 179-187, 2018. DOI: 10.1080/00029890.2017.1405685.

Problem 12025 is Proposed by Askat Dzhumadil'daev, S. Demirel University, Almaty, Kazakhstan.

2 Problem Statement

The Chebyshev polynomials of the second kind are defined by the recurrence relation

$$U_0(x) = 1, U_1(x) = 2x \text{ , and } U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \text{ for } n \geq 2. \tag{1}$$

For an integer n with $n \geq 2$, prove

$$\det \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & x^{n-4} & \dots & 0 & 1 \\ x^{n-1} & x^{n-2} & x^{n-3} & \dots & x & 0 \end{bmatrix} = (-1)^{n-1} x^{n/2} U_{n-2}(\sqrt{x}). \tag{2}$$

3 The Solution

We begin by denoting the left hand side of equation (2) by $D_n(x)$. That is,

$$D_n(x) := \det \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & x^{n-4} & \dots & 0 & 1 \\ x^{n-1} & x^{n-2} & x^{n-3} & \dots & x & 0 \end{bmatrix}. \tag{3}$$

We also define a closely related function $E_n(x)$ by

$$E_n(x) := \det \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & x^{n-4} & \dots & 0 & 1 \\ x^{n-1} & x^{n-2} & x^{n-3} & \dots & x & 0 \end{bmatrix}. \tag{4}$$

We record the first several cases:

$$D_1(x) = \begin{vmatrix} 0 \end{vmatrix} = 0,\ D_2(x) = \begin{vmatrix} 0 & 1 \\ x & 0 \end{vmatrix} = -x,\ D_3(x) = \begin{vmatrix} 0 & 1 & 1 \\ x & 0 & 1 \\ x^2 & x & 0 \end{vmatrix} = 2x^2, \tag{5}$$

and

$$E_1(x) = \begin{vmatrix} 1 \end{vmatrix} = 1,\ E_2(x) = \begin{vmatrix} 1 & 1 \\ x & 0 \end{vmatrix} = -x-1,\ E_3(x) = \begin{vmatrix} 1 & 1 & 1 \\ x & 0 & 1 \\ x^2 & x & 0 \end{vmatrix} = 2x^2 - x.$$

Now we find recurrence relations between the D_k and E_k.

Claim 1. *For $k \geq 1$,*

$$D_k(x) = -xD_{k-1}(x) - xE_{k-1}(x), \tag{6}$$
$$E_k(x) = (1-x)D_{k-1}(x) - xE_{k-1}(x). \tag{7}$$

Proof. First we manipulate $D_k(x)$ by subtracting the second row from the first. Thus

$$D_k(x) = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-2} & x^{k-3} & x^{k-4} & \dots & 0 & 1 \\ x^{k-1} & x^{k-2} & x^{k-3} & \dots & x & 0 \end{vmatrix} \overset{R1 \mapsto R1-R2}{=} \begin{vmatrix} -x & 1 & 0 & \dots & 0 & 0 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-2} & x^{k-3} & x^{k-4} & \dots & 0 & 1 \\ x^{k-1} & x^{k-2} & x^{k-3} & \dots & x & 0 \end{vmatrix}.$$

Now expand the determinant across the top row, obtaining

$$D_k(x) = (-x)\begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-3} & x^{k-4} & x^{k-5} & \dots & 0 & 1 \\ x^{k-2} & x^{k-3} & x^{k-4} & \dots & x & 0 \end{vmatrix} - (1)\begin{vmatrix} x & 1 & 1 & \dots & 1 & 1 \\ x^2 & 0 & 1 & \dots & 1 & 1 \\ x^3 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-2} & x^{k-4} & x^{k-5} & \dots & 0 & 1 \\ x^{k-1} & x^{k-3} & x^{k-4} & \dots & x & 0 \end{vmatrix}.$$

And by factoring out x from the first column of the second matrix, we have

$$- x\begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-3} & x^{k-4} & x^{k-5} & \dots & 0 & 1 \\ x^{k-2} & x^{k-3} & x^{k-4} & \dots & x & 0 \end{vmatrix} - x\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-3} & x^{k-4} & x^{k-5} & \dots & 0 & 1 \\ x^{k-2} & x^{k-3} & x^{k-4} & \dots & x & 0 \end{vmatrix}$$

$$= -xD_{k-1}(x) - xE_{k-1}(x).$$

This verifies the first claimed formula.

Now consider similar manipulations for $E_k(x)$. In particular, we have

$$E_k(x) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-2} & x^{k-3} & x^{k-4} & \dots & 0 & 1 \\ x^{k-1} & x^{k-2} & x^{k-3} & \dots & x & 0 \end{vmatrix} = \begin{vmatrix} 1-x & 1 & 0 & \dots & 0 & 0 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-2} & x^{k-3} & x^{k-4} & \dots & 0 & 1 \\ x^{k-1} & x^{k-2} & x^{k-3} & \dots & x & 0 \end{vmatrix},$$

and expanding the determinant across the top row gives

$$E_k(x) = (1-x)\begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 \\ x^2 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-3} & x^{k-4} & x^{k-5} & \dots & 0 & 1 \\ x^{k-2} & x^{k-3} & x^{k-4} & \dots & x & 0 \end{vmatrix} - (1)\begin{vmatrix} x & 1 & 1 & \dots & 1 & 1 \\ x^2 & 0 & 1 & \dots & 1 & 1 \\ x^3 & x & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{k-2} & x^{k-4} & x^{k-5} & \dots & 0 & 1 \\ x^{k-1} & x^{k-3} & x^{k-4} & \dots & x & 0 \end{vmatrix}.$$

Factoring out x from the first column of the second matrix then completes the proof of the second claimed formula. □

We next obtain a single recursive formula for the D_k.

Claim 2. *For $k \geq 2$,*

$$D_k(x) = -2xD_{k-1}(x) - xD_{k-2}(x).$$

Proof. Consider the formulas from Claim 1. Expanding (7) we have

$$E_k(x) = D_{k-1}(x) - xD_{k-1}(x) - xE_{k-1}(x),$$

and making use of (6) we therefore have

$$E_k(x) = D_{k-1}(x) + D_k(x).$$

This implies that

$$E_{k-1}(x) = D_{k-2}(x) + D_{k-1}(x).$$

Now substitute this result into (6):

$$\begin{aligned} D_k(x) &= -xD_{k-1}(x) - xE_{k-1}(x) \\ &= -xD_{k-1}(x) - x[D_{k-2}(x) + D_{k-1}(x)] \\ &= -2xD_{k-1}(x) - xD_{k-2}(x). \end{aligned}$$

Thus we have the recursive formula. □

Finally, to complete the proof we need to show $D_n(x) = (-1)^{n-1}x^{n/2}U_{n-2}(\sqrt{x})$ for $n \geq 2$. This will be proven by strong induction.

Base cases. The cases $n = 2$ and $n = 3$ follow from (5) and the initial Chebyshev polynomials $U_0(x) = 1$ and $U_1(x) = 2x$. In particular,

$$\begin{aligned} D_2(x) &= -x = -x \cdot U_0(\sqrt{x}), \\ D_3(x) &= 2x^2 = x^{\frac{3}{2}} \cdot U_1(\sqrt{x}). \end{aligned}$$

(Strong) Inductive Step. Now assume that $D_n(x) = (-1)^{n-1}x^{n/2}U_{n-2}(\sqrt{x})$ for $2 \leq n \leq k$. Let $n = k+1$ and consider $D_{k+1}(x^2)$. By Claim 2 and the inductive assumption we have

$$\begin{aligned} D_{k+1}(x^2) &= -2x^2D_k(x^2) - x^2D_{k-1}(x^2) \\ &= -2x^2[(-1)^{k-1}x^kU_{k-2}(x)] - x^2[(-1)^{(k-1)-1}x^{k-1}U_{(k-1)-2}(x)] \\ &= 2(-1)^kx^{k+2}U_{k-2}(x) + (-1)^{k+1}x^{k+1}U_{k-3}(x) \\ &= (-1)^kx^{k+1}[2xU_{k-2}(x) - U_{k-3}(x)] \\ &= (-1)^kx^{k+1}[U_{k-1}(x)]. \end{aligned}$$

The last equality follows from the recurrence relation for the Chebyshev polynomials.

Replacing x^2 by x shows that

$$D_{k+1}(x) = (-1)^kx^{(k+1)/2}U_{k-1}(\sqrt{x}).$$

This completes the inductive step, and thus

$$D_n(x) = (-1)^{n-1}x^{n/2}U_{n-2}(\sqrt{x}) \text{ for all } n \geq 2.$$