American Math. Monthly Problem 12008

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Problem Statement

You hold in your hand a deck of n cards, numbered 1 to n from top to bottom. Shuffle them as follows. Put the top card in the deck on the bottom and the second card on the table. Repeat this step until all the cards are on the table.

Part (a)

For which n does card number 1 end up at the top of the deck of cards on the table?

The answer is all n of the form $n = 2^j,\ j \geq 0$. In other words, card 1 finishes on top after one shuffle if and only if n is a power of two.

To approach this problem, we characterize the final position of card 1 after a single shuffle. Let $p(n)$ denote this position, counting from the top (so that initially card 1 is in position 1). In order to do this, we express n in the following form

$$n = 2^j q = 2^j(2k+1) \tag{1}$$

where j is as large as possible and $q = 2k+1$ is odd.

If n is even (so $j \geq 1$), then during the first "pass" through the deck, all of the even labeled cards are taken out and we are left with card 1 on top, proceeded by all of the odd labeled cards. We can now imagine that we relabel the cards in the deck by replacing $2c-1$ by c, giving new labels 1 to $\frac{n}{2} = 2^{j-1}m$ as follows

$$1,\ 3,\ 5,\ \ldots,\ 2^j q - 3,\ 2^j q - 1 \mapsto 1,\ 2,\ 3,\ \ldots,\ 2^{j-1}q - 1,\ 2^{j-1}q.$$

Thus if n is even, $p(n) = f\left(\frac{n}{2}\right)$. This process therefore repeats j times until an odd number of cards remain, leaving $p(n) = p(q)$.

Now consider a deck with an odd number of cards, $n = 2k+1$. The shuffling process now takes out the k even cards until the remaining deck has card $2k+1$ on top, followed by card 1. Card 1 is then the next card to be placed on the table, followed by the remaining k. Thus, the final position of card 1 is

$$p(n) = k + 1 = \frac{n+1}{2}. \tag{2}$$

For card 1 to finish the shuffle on top requires $k+1 = 1$, or $k = 0$. This implies that n is of the form

$$n = 2^j \cdot (0+1) = 2^j,$$

as claimed.

Part (b)

Shuffle the deck a second time in the same way. For which n does card number 1 end up at the top of the cards on the table?

The answer is all n of the form

$$n = 2^j \left(\frac{2^{(j+2)(t+1)} - 1}{2^{j+2} - 1} \right) = 2^j \left(2^{(j+2)t} + \cdots + 2^{j+2} + 1 \right), \tag{3}$$

where $j, t \geq 0$. This is a *complete* characterization: card 1 is on top after two shuffles if and only if n has the given form. The following list contains all such values of n up to 1000:

$$1,\ 2,\ 4,\ 5,\ 8,\ 16,\ 18, 21,\ 32,\ 64,\ 68,\ 85,\ 128,\ 146,\ 256,\ 264,\ 341,\ 512.$$

Our method for approaching this problem is similar to that of Part (a). We will use our result for the value of $p(n)$. Since we are doing two shuffles, however, in addition to characterizing where card one lands after the first shuffle, we also wish to characterize which card rests on top of the deck after the first shuffle.

Fortunately, this corresponds to the solution of the well-known Josephus problem. If we arrange the cards in the original deck in a circle, then our shuffling method removes cards from the deck in exactly the same way as soldiers are removed from the circle in the Josephus problem (minus the stabbing). The last remaining soldier in the circle corresponds to the last card placed on the table, so the position of the "winning" soldier gives us the label of top card, which we denote by $w(n)$. To find this, we first express n in the form

$$n = 2^m + \ell, \tag{4}$$

where $\ell < 2^m$. The result of the Josephus problem (see equation (22) of [1], for example) states that

$$w(n) = 2\ell + 1 \tag{5}$$

So after the first shuffle, card one is in position $p(n)$, and we know that given a deck of n cards, the card that begins in position $w(n)$ will finish on top. Therefore, for two shuffles card number 1 finishes on top when $p(n) = w(n)$. We can then combine equations (2) and (5), arriving at the condition

$$k + 1 = 2\ell + 1,$$

so $k = 2\ell$. We should also equate our two expressions of n, (1) and (4), which gives

$$2^m + \ell = 2^j(2k + 1) = 2^j(4\ell + 1).$$

Solving for ℓ, we arrive at

$$\ell = \frac{2^m - 2^j}{2^{j+2} - 1}. \tag{6}$$

Since ℓ must be an integer, we find that

$$\left(2^{j+2} - 1\right) \mid 2^j \left(2^{m'} - 1\right), \tag{7}$$

where $m' := m - j$. Note that the first term is necessarily odd, so this is equivalent to the requirement that

$$\left(2^{j+2} - 1\right) \mid \left(2^{m'} - 1\right). \tag{8}$$

Lemma 3.1 of [2] implies that for $a, b \in \mathbb{N}$, $(2^a - 1) \mid \left(2^b - 1\right)$ if and only if $a \mid b$[1] Using this, we arrive at our final result that card number 1 finishes on the top of the deck after two shuffles if and only if $(j+2) \mid m'$. This completes the proof.

[1] This Lemma states that the least positive residue of $2^a - 1$ modulo $2^b - 1$ is $2^r - 1$ where r is the least positive residue of a modulo b. The claim follows as the case $r = 0$.

Bibliography

[1] L. Halbeisen, *The Josephus problem*, Journal de Theorie des Nombres de Bordeaux **9** (1997) 303–318.

[2] K. Rosen, *Elementary Number Theory and its Applications*, Addison-Wesley.