LSU Problem Solving Seminar - Fall 2019
Oct. 30: Post-Exam Review of Virginia Tech Math Contest

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This week's practice sheet provides a detailed look at the problems from last weekend's 2019 Virginia Tech Regional Math Contest. Each contest problem is **preceded** by a related problem that illustrates some relevant techniques in a simpler context.

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1. For a positive integer m, denote the sum of its decimal digits by $\mathrm{Dig}(m)$. For example, $\mathrm{Dig}(297) = 18$.
   (a) Prove that $m - \mathrm{Dig}(m)$ is always a non-negative multiple of 9.
   (b) Suppose that m is a multiple of 11. Show that $\mathrm{Dig}(m)$ **cannot** be equal to $1, 3, 5, 7$, or 9. Find such an m where $\mathrm{Dig}(m) = 11$.
   (c) If m is a multiple of 99, prove that $\mathrm{Dig}(m) \geq 18$.

2. [VTRMC **2019 #1**] For each positive integer n, define $f(n)$ to be the sum of the digits of 2771^n (so $f(1) = 17$). Find the minimum value of $f(n)$ (where $n \geq 1$). Justify your answer.

3. Let ABC be a triangle. Suppose that X is a point on the edge AB such that $|AX| = \alpha|AB|$ (note that $\alpha \leq 1$). Furthermore, P and Q are points on AC such that $|PQ| = \beta|AC|$. Determine the ratio of the area of PQX to the area of ABC.

4. [VTRMC **2019 #2**] Let X be the point on the side AB of the triangle ABC such that $BX/XA = 9$. Let M be the midpoint of BX and let Y be the point on BC such that $\angle BMY = 90°$. Suppose AC has length 20 and that the area of the triangle XYC is $9/100$ of the area of the triangle ABC. Find the length of BC.

5. (a) Show that if $f(x)$ has two real roots, then $f'(x)$ has at least one real root.
   (b) Prove that $g(x) = 7x^6 - 6x^5 + 5x^4 - 4x^3 + 3x^2 - 4x + 1$ has a real root.

6. [VTRMC **2019 #3**] Let n be a nonnegative integer and let $f(x) = a_n x^n + a_{n-1}x^{n-2} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial with real coefficients a_i. Suppose that
$$\frac{a_n}{(n+1)(n+2)} + \frac{a_{n-1}}{n(n+1)} + \cdots + \frac{a_1}{6} + \frac{a_0}{2} = 0.$$
   Prove that $f(x)$ has a real zero.

7. (a) Evaluate $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx.$

   Hint: Make a change of variables to show that $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx$. *Now add the two expressions. . . .*

(b) Suppose that $f(x)$ is a function such that $f(x) \neq f(-x)$ for all x (in particular, $f(0) \neq 0$). Show that if $a \geq 0$,

$$\int_{-a}^{a} \frac{f(x)}{f(x)+f(-x)}\,dx = a.$$

8. [VTRMC **2019 #4**] Compute $\displaystyle\int_0^1 \frac{x^2}{x+\sqrt{1-x^2}}\,dx.$

9. (a) Find the general solution to the differential equation

$$y'(x) + 2xy(x) = 0.$$

Hint: If you multiply by the "integrating factor" e^{x^2}, you should be able to simplify the left side using the product rule....

(b) Find the solution to the differential equation

$$x^2y'(x) + y(x) = 0,$$

such that $f(1) = 1$.

10. [VTRMC **2019 #5**] Find the general solution of the differential equation

$$x^4\frac{d^2y}{dx^2} + 2x^2\frac{dy}{dx} + (1-2x)y = 0,$$

valid for $0 < x < \infty$.

11. (a) The real numbers satisfy the *Archimedean property:* For any real number x, there is an integer $n > x$ (this follows from the simple fact that the integers are unbounded). Prove that this is equivalent to the statement that for any real $\varepsilon > 0$, there is a positive integer n such that $\frac{1}{n} < \varepsilon$.

(b) Prove that any real interval (x, y) contains a rational number.

12. [VTRMC **2019 #6**] Let S be a subset of $\mathbb{R}$ with the property that for every $s \in S$, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \cap S = \{s\}$. Prove there exists a function $f : S \to \mathbb{N}$, the positive integers, such that for all $s, t \in S$, if $s \neq t$, then $f(s) \neq f(t)$.

13. The *Harmonic series* is the sum $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots$.

(a) It is known that the harmonic series diverges – try to prove this as many different ways as you can!

Hint: 1. *Integral comparison;* 2. *Grouping: Take the 1st term, then the 2nd term, then the 3rd and 4th, then 5th–8th, etc.;* 3. *Consider* $e^1e^{\frac{1}{2}}e^{\frac{1}{3}}\cdots$..

(b) Does the series $1 + \dfrac{1}{101} + \dfrac{1}{201} + \dfrac{1}{301} + \cdots$ converge?

(c) Does the series $1 + \dfrac{1}{11} + \dfrac{1}{101} + \dfrac{1}{1001} + \cdots$ converge?

14. [VTRMC **2019 #7**] Let S denote the positive integers that have no 0 in their decimal expansion. Determine whether $\displaystyle\sum_{n\in S} n^{-99/100}$ converges.

Answers for numerical problems. **#1**: 17 **#2**: 20, **#4**: $\frac{1}{4}$, **#5**: $(ax+b)e^{1/x}$, **#7**: Yes.