LSU Problem Solving Seminar - Fall 2019
Nov. 6: Polynomials and Complex Numbers

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ be a polynomial with real coefficients. It is *monic* if the leading coefficient $a_n = 1$. The *degree* of a polynomial is the exponent of the leading term, in this case n. A *root* of f is a value r such that $f(r) = 0$.

Useful facts and strategies:

- **Rational Roots Test.** If all of the a_i are integers and $r = \frac{p}{q}$ is a root, then p is a divisor of a_0 and q is a divisor of a_n.
- **Descartes' Rule of Signs.** If the non-zero coefficients of $f(x)$ change sign s times, then f has at most s positive roots (with multiplicity). The actual number of positive roots is less than s by some multiple of 2. Replacing x by $-x$ gives a similar test for negative roots.
- **Polynomial Division Algorithm.** A polynomial $f(x)$ is a *multiple* of $g(x)$ if $f(x) = h(x) \cdot g(x)$ for some polynomial $h(x)$. If $f(x)$ is not a multiple of $g(x)$, then there are polynomials $q(x)$ ("quotient") and $r(x)$ ("remainder") such that $f(x) = q(x) \cdot g(x) + r(x)$, where $r(x)$ has lower degree than $g(x)$.
- **Repeated Roots.** A polynomial $f(x)$ is divisible by $(x - r)^k$ (i.e. the root r has *multiplicity* at least k) if and only if $f(r) = 0, f'(r) = 0, \ldots, f^{(k-1)}(r) = 0$.
- **Fundamental Theorem of Algebra.** A polynomial of degree n has exactly n complex roots, counted with multiplicity. In particular, it has at most n real roots. Furthermore, if the roots are $r_1, \ldots, r_n$, then $f(x) = c(x - r_1) \cdots (x - r_n)$ for some constant c.
- **Sum and Product of Roots.** If a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has roots (with repetition) $r_1, \ldots, r_n$, then
$$a_{n-1} = -(r_1 + \cdots + r_n); \qquad a_0 = (-1)^n r_1 \cdots r_n.$$
- **Roots of Unity.** The (complex) roots of $x^n - 1$ are $1, e^{\frac{2\pi i}{n}}, e^{\frac{2 \cdot 2\pi i}{n}}, \ldots, e^{\frac{(n-1) \cdot 2\pi i}{n}}$. These can also be written as $1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}$, where $\zeta_n := e^{\frac{2\pi i}{n}}$. The previous property implies that
$$1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1} = 0.$$

Warm Up

---

1. Find the factorization of the following polynomials (with real coefficients):
   (a) $x^2 - x - 1$,
   (b) $x^3 - \dfrac{3}{2}x^2 - \dfrac{5}{2}x + 3$,
   (c) $x^5 - 3x^3 + 2x$.
2. The polynomial $p(x) = x^4 - x^3 - 7x^2 + x + 6$ has four distinct rational roots, one of which is $x = -2$. Find the other roots and complete factorization of $p(x)$.

3. (a) If $p(x)$ is a polynomial of degree k, show that $p(x+1) - p(x)$ is a polynomial of degree $k-1$.
   (b) Find all polynomials that satisfy
$$p(x+1) + p(x-1) = 2p(x) + 2 \qquad \text{for all } x.$$

Main Problems

---

4. Let $p(x) = x^4 + 4$.
   (a) Show that $p(x)$ is not divisible by $x - r$ for any real r.
   (b) Find the factorization of $p(x)$.
   Hint: All coefficients in the factorization are integers.

5. Let $\alpha := \sqrt[3]{9+4\sqrt{5}} + \sqrt[3]{9-4\sqrt{5}}$. It is a fact that α is equal to an integer; determine which one!
   Hint: Calculate α^3.

6. For a function $f(x)$, let $f^n(x)$ denote the function iterated n times, i.e. $f(f(\cdots(f(x))\cdots))$.
   (a) Let $f(x) = x^2 + 2x + 1$. Show that $f(x)$ has a real root, but $f^2(x)$ has no real roots.
   (b) Consider the polynomial $p(x) = x^2 - 10x + 10$. Does $p^{10}(x)$ have a real root?

7. (a) [**Gelca-Andreescu 170**] Let x, y, z be positive integers greater than 1. Prove that the expression
$$(x+y+z)^3 - (-x+y+z)^3 - (x-y+z)^3 - (x+y-z)^3$$
   is the product of seven (not necessarily distinct) integers each of which is greater than 1.
   Hint: What happens if you plug in $x = 0$?
   (b) [**Gelca-Andreescu 171**] Factor completely the expression
$$(x+y+z)^5 - (-x+y+z)^5 - (x-y+z)^5 - (x+y-z)^5.$$

8. Let $p_n(x) := x^3 - x + n$, where n is an integer.
   (a) Show that if $p_n(x)$ divides a polynomial $f(x)$, then $f(-1), f(0)$, and $f(1)$ are all multiples of n.
   (b) Find the unique n such that that $p_n(x)$ divides
$$q(x) = x^8 - 5x^2 - 6x - 8.$$

9. [Putnam **1963 B1**] Find all integers n such that $x^2 - x + n$ divides $x^{13} + x + 90$.

10. [Putnam **1990 B5**] Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial
$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
   has exactly n distinct real roots?