LSU Problem Solving Seminar - Fall 2019
Nov. 13: Inequalities

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Useful facts:

- **Arithmetic-Geometric Mean Inequality.** If $a_1, \ldots, a_n$ are non-negative real numbers, then
$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$
Furthermore, the right side is strictly larger than the left unless all of the a_i are equal.
- **Hölder's p-norm Inequality.** If $0 < p < q$ and $a_1, \ldots, a_n$ are non-negative real numbers, then
$$\left(\frac{a_1^p + \cdots + a_n^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{a_1^q + \cdots + a_n^q}{n}\right)^{\frac{1}{q}},$$
with strict inequality unless all of the a_i are equal.
- **Cauchy-Schwarz Inequality.** If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then
$$\left(a_1b_1 + \cdots + a_nb_n\right)^2 \leq \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right).$$
Furthermore, the right side is strictly larger unless $(b_1, \ldots, b_n) = (\lambda a_1, \ldots, \lambda a_n)$ for some real λ. Written in vector notation and Euclidean distance, $|\overrightarrow{a} \cdot \overrightarrow{b}|^2 \leq |\overrightarrow{a}|^2 \cdot |\overrightarrow{b}|^2$.
- **Triangle Inequality.** If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then
$$\sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}.$$
Written in vector notation, $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.
- **Rearrangement Inequality.** Suppose that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. If $a_1', a_2', \ldots, a_n'$ is any reordering of $a_1, \ldots, a_n$, then
$$a_1'b_1 + a_2'b_2 + \cdots + a_n'b_n < a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Warm Up

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1. For each of the following pairs, determine which expression is larger (without using a calculator!):
   (a) $\left(1 + \frac{1}{10}\right)^{10}$ or $\left(1 + \frac{1}{2019}\right)^{2019}$?
   (b) $2^9 3^8 4^7 5^6$ or $6^5 7^4 8^3 9^2$?
   (c) $2019!$ or 1019^{2019}?
2. Find the minimum value over all positive x of $x^3 + x^2 - 2 + \frac{1}{x} + \frac{1}{x^4}$.
   Hint: Try to do this without calculus!

3. Determine which is larger:

$$1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{16}+\sqrt{17} \qquad \text{or} \qquad 51?$$

Hint: Use the Cauchy-Schwarz inequality.

Main Problems

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4. Suppose that $a_1, \dots, a_n$ are the integers $1, 2, \dots, n$ in some order (i.e., a *permutation*). What is the minimum possible value of

$$\frac{1}{a_1}+\frac{2}{a_2}+\cdots+\frac{n}{a_n}?$$

5. (a) A polynomial is partially erased on the board, leaving only the following coefficients:

$$p(x)=x^7-7x^6+\cdots\ ?\ ?\ ?\ ?\ ?\ \cdots-1.$$

There is also a sentence fragment below that states: "*. . . since all roots of $p(x)$ are positive real numbers.*"

Determine $p(x)$.

(b) The fact that the roots in part (a) were positive is essential! Find two different polynomials of the form $f(x)=x^3-3x^2+ax-1$ that factor into real roots.

6. Suppose that x and y are positive, and let $f(x,y)=\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2$.

(a) Find the minimum value of $f(x,y)$ if $x+y=1$.

(b) Find the minimum value of $f(x,y)$ if $x+y=2$.

7. (a) Suppose that $a<b$ and $c<d$. Prove that $ad+bc<ac+bd$.

Remark: This is the first case of the Rearrangement Inequality.

(b) If $x\in(0,\frac{\pi}{2})$, determine the minimum value of $\dfrac{\sin^3 x}{\cos x}+\dfrac{\cos^3 x}{\sin x}$.

8. (a) Find all positive integer solutions to

$$a_1+\cdots+a_{10}=50 \qquad \text{and} \qquad a_1^2+\cdots+a_{10}^2=262.$$

Hint: Noting that the average of the a_k's is 5, consider $\displaystyle\sum_{k=1}^{10}(a_k-5)^2$.

(b) [**Gelca-Andreescu 107**] Let $a_1, a_2, \dots, a_n$ be real numbers such that

$$a_1+a_2+\cdots+a_n\ge n^2 \qquad \text{and} \qquad a_1^2+a_2^2+\cdots+a_n^2\le n^3+1.$$

Prove that $n-1\le a_k\le n+1$ for all k.

9. [Putnam **2003 B2**] Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \dots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n<2/n$.