Important upcoming dates:

- The Problem-Solving Seminar will **not** meet on Wednesday, Nov. 27 due to the Thanksgiving holiday. The last meeting of the semester will be Wednesday, Dec. 4.
- Putnam Mathematical Competition, **Sat., Dec. 7.** The Exam will take place in Lockett 232 from 8:30 A.M. – 5:00 P.M.

LSU Problem Solving Seminar - Fall 2019
Nov. 20: Integration

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Useful facts:

- **Partial Fractions.** If $f(x)$ is a polynomial whose degree is less than n, then there are constants $a_1, \ldots, a_n$ such that
$$\frac{f(x)}{(x-r_1)\cdots(x-r_n)} = \frac{a_1}{x-r_1} + \cdots + \frac{a_n}{x-r_n}.$$
(Here the roots r_i must be distinct – there is a more complicated version for repeated roots.)
- **Fundamental Theorem(s) of Calculus.** Suppose that $f(x)$ is a continuous function.
  - If $F(x)$ is an antiderivative of f, then $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$.
  - Define $g(x) := \displaystyle\int_a^x f(t)dt$. Then $g'(x) = f(x)$.
- **Integration By Parts.** Suppose that f and g are differentiable. Then
$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx.$$
- **Substitution.**
$$\int_{x=a}^{b} f(u(x))u'(x)dx = \int_{u=u(a)}^{u(b)} f(u)du.$$
- **Symmetries and Substitution.** Always remember that integration problems on Mathematics Contests are meant to have solutions! A complicated integral often has a hidden symmetry or substitution that makes it much easier to evaluate. For example, if $f(x)$ is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0$.

Warm Up

1. Find the following antiderivatives.

(a) $\displaystyle\int \frac{x}{\cos^2(x^2)}\,dx.$

(b) $\displaystyle\int \frac{1}{x^3 - x}\,dx.$

2. Evaluate the following integrals with as **little** computation as possible.

(a) $\displaystyle\int_{-2}^{2} \frac{x^2}{\sinh(x)}\,dx.$

(b) $\displaystyle\int_{0}^{1} 1 + \sqrt{1 - x^2}\,dx.$

Hint: If you write $y = 1 + \sqrt{1 - x^2}$*, what sort of figure does the equation in* x *and* y *describe?*

3. (a) Find the antiderivative $\displaystyle\int \frac{1}{x \ln x}\,dx.$

Hint: Let $u = \ln x$.

(b) Calculate the derivative of $f(x) = \ln(\ln(x^2))$. Does this contradict your answer to part (a)? Explain.

Main Problems

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4. (a) Evaluate (through direct calculation)

$$\int_0^2 \frac{x^3}{4} + \sqrt[3]{4x}\,dx.$$

(b) Find a graph and/or substitution that easily explains your answer from part (a).

(c) Evaluate

$$\int_0^{\frac{\pi}{2}} 2^{\sin x} + \frac{2}{\pi} \arcsin\left(\log_2\left(\frac{2x}{\pi} + 1\right)\right)\,dx.$$

5. In this problem you will evaluate

$$I := \int_0^{\pi} \log(\sin u)\,du.$$

(a) Show that

$$\frac{I}{2} = \int_0^{\frac{\pi}{2}} \log(\sin u)\,du = \int_0^{\frac{\pi}{2}} \log(\cos u)\,du.$$

(b) Combine the two expressions from part (a) in such a way that you can use the sine double angle formula. After doing so, you should be able to solve for I.

(c) Now evaluate

$$\int_0^{\pi} \log(1 + \cos u)\,du.$$

6. (a) Find the antiderivative

$$\int \frac{x^4 - x^3 + 1}{x^4 + 1}\, dx.$$

(b) [**Gelca-Andreescu 544**] For a positive integer n, compute the integral

$$\int \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}\, dx.$$

7. [Putnam **1968 A1**] Show that

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1}\, dx = \frac{22}{7} - \pi.$$

8. In this problem you will learn to use *Leibniz's Method* (also popularized more recently as *Feynman's Method*) for integration.

(a) For $a \geq 0$, let

$$I(a) := \int_0^\infty \log(a^2 + x^2)\, dx.$$

Calculate $I'(a)$.

Hint: You will need to know the derivative of the inverse tangent function (arctan).

(b) Now use the Fundamental Theorem of Calculus to evaluate

$$\int_0^\infty \left(\log(1 + x^2) - \log(x^2)\right)\, dx.$$

9. [Putnam **1982 A3**] Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx.$$