LSU Problem Solving Seminar - Fall 2019
Dec. 4: Putnam Review / Miscellaneous

Thanks to Farid Bouya *for preparing part of this week's problem sheet!*

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Putnam Mathematical Competition, Sat., Dec. 7
Lockett Hall 232, 8:30 A.M. – 5:00 P.M.

Test-taking tips:

- **Format.** The Exam is given in two 3-hour sessions of 6 problems each, with a lunch break from 12:00 – 2:00 P.M. The morning session's problems are labeled **A1** – **A6**, and the afternoon's **B1** – **B6**.
- **Grading.** Each problem is graded out of **10** points, for a maximum possible score of 120. Typically there is very little partial credit given, and a submitted problem will receive **0, 1, 2, 9,** or **10** points.
- **A1/A2/B1.** In recent years these three problems have been the "easiest" part of the exam. More generally, the problems in each session are roughly ordered by difficulty. This is not an absolute rule, but you should expect that **A1** will have a relatively short solution, whereas **A6** may not. You should devote at least 15 minutes each to **A1, A2, B1** before moving on to the rest of the Exam.
- **1 hour per write-up.** In order to get full credit, your solutions must be written very carefully. If you use a result from a course, refer to it by name (e.g. Fundamental Theorem of Calculus). After you solve a problem, you should plan on spending approximately one hour writing your solution. In light of the grading described above, it is better to solve one problem completely than several problems partially.

Warm Up

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1. A positive integer is placed in every square of an infinite chess board in such a way that every number is the mean of its four neighbors. Show that all the numbers are equal.

   Hint: The well-ordering principle *states that every non-empty set of positive integers contains a least element.*

2. A tournament consists of n players, and every pair of players play exactly once, with one player winning. Show that every tournament has a Champion, which is a player C such that every other player P has either
   - lost to C, or
   - lost to some player M, where M lost to C.

   Hint: Consider the player with the most wins. If she lost to P, what can be said about the players that P lost to?

The remainder of this week's practice sheet provides a detailed look at several problems from previous Putnam Competitions. Most of these problems are **preceded** by one or more related problems that illustrate some relevant concepts in a simpler context.

3. [Putnam **1992 A1**] Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

   (i) $f(f(n)) = n$ for all integers n;

   (ii) $f(f(n+2)+2) = n$ for all integers n;

   (iii) $f(0) = 1$.

4. In the country of Odd, the parliament has n members $x_1, x_2, \ldots, x_n$. It is known that x_i has $2k_i - 1$ enemies in the parliament, where $k_i \geq 1$ (the enemy relationship is symmetric, so that x_i is x_j's enemy if and only if x_j is x_i's enemy). Show that the parliament can be partitioned into two groups such that the members of each group have less than half of their original number of enemies (so that x_i has at most $k_i - 1$ enemies in its group).

   Hint: Consider the partition with the **least** *number of enemy pairs.*

5. [Putnam **2017 A4**] A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be partitioned into two groups of N students in such a way that the average score for each group is exactly 7.4.

6. (a) Let ABC be a triangle and let X be the midpoint of AB, Y be the midpoint of AC, and Z be the midpoint of BC. Show that

   $$\triangle AXY = \triangle BXZ = \triangle CYZ = \triangle XYZ$$

   (this notation means the triangles are *congruent*).

   (b) Let XYZ be a triangle, let l_X be the line parallel to YZ passing X, l_Y be the line parallel to XZ passing Y, and l_Z be the line parallel to XY passing Z. Let A be the intersection of l_X and l_Y, B be the intersection of l_X and l_Z, and C be the intersection of l_Y and l_Z. Show that

   $$\triangle AXY = \triangle BXZ = \triangle CYZ = \triangle XYZ.$$

7. [Putnam **2016 B3**] Suppose that S is a finite set of points in the plane such that the area of triangle $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

8. The generalized Binomial Theorem states that for real α,

   $$(1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{k} x^k,$$

   where the *generalized Binomial Coefficients* are

   $$\binom{\alpha}{k} := \frac{(\alpha)_k}{k!}, \qquad \text{with} \qquad (\alpha)_k := \alpha(\alpha-1)\cdots(\alpha-k+1).$$

(a) Let $f(x) := (1+x)^\alpha$. Prove the generalized Binomial Theorem by showing that $f^{(k)}(0) = (\alpha)_k$, and then using Taylor's Theorem.

(b) Show that

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \cdots = \sum_{k\geq 0} \frac{(k+1)(k+2)}{2} x^k.$$

(c) Let $g(x) := (x)_k$. Prove that

$$g'(x) = (x)_k \cdot \left(\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x+k-1} \right).$$

9. [Putnam **1992 A2**] Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

10. A *slice* of the plane is the set of points strictly between two parallel lines. The *width* of a slice is the distance between the lines.

    (a) Suppose that $\{S_n\}_{n\geq 1}$ are all slices defined by horizontal lines in the plane (i.e., each S_n is a slice consisting of points (x, y) satisfying $a_n < y < b_n$ for some constants a_n, b_n). Show that if the slices cover the plane, then the sum of the widths must diverge.

    (b) Suppose that $\{S_n\}_{n\geq 1}$ are slices with widths $\{w_n\}$, such that $\sum_{n\geq 1} w_n < \infty$. Prove that these slices **cannot** cover the plane.

    Hint: Consider the intersection of the slices with a very large disc.

11. [Putnam **1975 B2**] A *slab* is the set of points strictly between two parallel planes. Prove that a countable sequence of slabs, the sum of whose thicknesses converges, cannot fill space.