- Virginia Tech Mathematics Contest. Sat., Oct. 26. **Sign-up deadline: Sep. 27**.
- Putnam Mathematical Competition. Sat., Dec. 7. **Sign-up deadline: Nov. 1**.

LSU Problem Solving Seminar - Fall 2019
Sep. 4: Classic Puzzles

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

Warm Up

1. How can 12 flags be hung around a square building so that exactly 4 flags are visible on each side?

2. (**Magic Triangle.**) Fill in the empty circles in Figure 1 with the digits 4 – 9 such that the sum on each side is the same.
   (a) What is the common value of those sums?
   (b) How many distinct solutions are there?

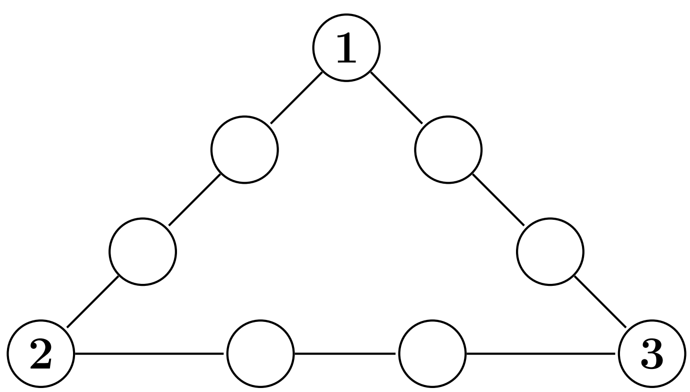

Figure 1: Shape for Magic Triangle.

3. (a) A farmer has grown a new variety of grapes that are 99% water by weight, and has seen massive consumer demand after advertising them as the "World's Juiciest"$^{\mathrm{TM}}$. After this season's harvest he therefore attempts to dry some of them so that he can also begin selling the "World's Juiciest" raisins. After a couple of days of drying in the sun, he receives a call from his Agricultural Chemist, who informs him that the grapes are still 95% water by weight. Furious about the lack of progress, he calls his Field Manager, who replies: "What more do you want from me? We started with 1000 pounds, and I'm sure it's much less now!"
   Should the farmer be upset? What is the current weight of the grapes?

(b) The farmer was planning on halting the drying process and packaging the raisins once the water content reaches 33%. How many pounds of raisins will he have in the end?

(c) The farmer's grapes were quite rare indeed – typical grape varieties are approximately 80% water by weight. Raisins are commonly produced so that only 20% of the weight is due to water. How many pounds of such raisins will be produced from 1000 pounds of (typical) grapes?

Remark: This problem explains why the most common dried fruits (grapes, figs, plums, cherries) are those whose water content is relatively low. Melons and berries can be as much as 95% water!

Main Problems

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4. (**Magic Triangle II.**) Recalling Problem 2, if the entries in the corners are not specified in advance, there are additional possible magic triangles.

   (a) Find a Magic Triangle where the sums along the sides are 23.

   (b) Find as many other examples as you can – try to classify them completely!

5. (**Three Links Problem.**)

   (a) A customer walks into a blacksmith's shop and lays a set of 5 triple links on the table (each link is of the shape seen in Figure 2), explaining that he would like them joined into a single chain of 15 links. The blacksmith replies that the cost will be 40 coins, since he charges 10 coins per link that must be cut, and the customer's request will require him to join the first triple link to the second, the second to the third, and so on, for a total of four links cut and reconnected.

   However, the customer screams in outraged reply: "Swindler! How dare you overcharge me! The cost should only be 30 coins." What does the customer have in mind – how can the blacksmith create a single chain with only 3 cuts?

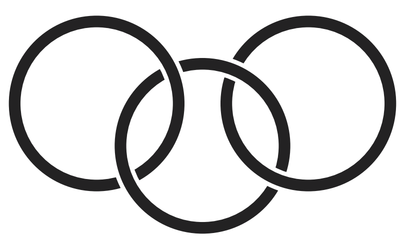

Figure 2: One of the triple links in Problem 5.

   (b) The customer returns a week later, and places six single links on the counter. Before he can explain his order, the blacksmith excitedly greets him: "You'll be happy to know that you've helped revolutionize our business! In fact, the new shop motto is 'Cut Once, Link Twice!' For example, whereas before I would have used five cuts to create a chain from those six links, adding each link to the end one at a time, now I'll only charge you 30 coins. I can create a triple link by cutting only the central link, and once I've done that twice, I'll join the two triple links with a third and final final cut. In fact, even if you had seven links, the price to make a chain would still only be 30 coins!"

What is the procedure for seven links? What is the minimum number of cuts required to create a chain of n links?

(c) The customer replies: "That's good to hear, but I would actually like a necklace made from these six links. How much will that be?"

"40 coins – it's just one more cut to join the ends of the chain."

"Dishonest as ever, I see! A six-link necklace should only be 30 coins."

What procedure does the customer have in mind this time?

6. (**Two/Three Pail Problem.**)

(a) A Scientist is preparing a cleaning solution that requires **exactly** 4 gallons of water. However, when she arrives at the river, she realizes that she only brought two buckets, which instead hold 3 and 5 gallons. How can she measure the required amount?

(b) After the first experiment's success, she returns to the river with several 8 gallon buckets, which she fills and returns to the lab. Explain how your solution to part (a) can be applied, using the full 8 gallon bucket and empty 3 and 5 gallon buckets to obtain exactly 4 gallons.

In fact, show that she can precisely obtain any whole number of gallons from 1 to 8.

(c) Later, the Scientist visits another laboratory, and asks for exactly 3 gallons of water. Having heard about the Scientist's successes, the Assistant in this lab produces three buckets that hold 4, 6, and 8 gallons. However, before he can begin, the Scientist says: "You fool! You'll never be able to achieve 3 gallons with those buckets." Why not?

(d) The Scientist continues: "Here – I've filled the 6 gallon bucket, and replaced your 8 gallon bucket with a 5 gallon bucket. You should be able to get me 3 gallons with this set." Is she correct? Can the assistant use a filled 6 gallon bucket and empty 4 and 5 gallon buckets in order to measure 3 gallons?

7. [VTRMC **1986 #3**] Express $\sinh(3x)$ as a polynomial in $\sinh x$. As an example, the identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ can be expressed as a polynomial in $\cos x$. (Recall that sinh denotes the hyperbolic sine defined by $\sinh x = \frac{e^x - e^{-x}}{2}$).

8. [Putnam **1993 B2**] Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to teach of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?