- Virginia Tech Mathematics Contest. Sat., Oct. 26. **Sign-up deadline: Sep. 27**.
- Putnam Mathematical Competition. Sat., Dec. 7. **Sign-up deadline: Nov. 1**.

# LSU Problem Solving Seminar - Fall 2019
## Sep. 11: Enumeration

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

Useful facts and strategies: ($n$ and $k$ are non-negative integers)

- **Pigeonhole Principle.** If more than $n$ elements are distributed among $n$ categories, then there will be a category with more than one element.
- **Permutations.** The number of ordered lists of $k$ distinct elements chosen from a set of $n$ objects is $P(n,k) := \frac{n!}{(n-k)!}$.
- **Binomial Coefficients.** Given two non-negative integers $n$ and $k$, the number of ways of choosing $k$ (unordered) objects from a set of $n$ is $C(n,k) = \binom{n}{k} := \frac{n!}{k!(n-k)!}$ (this is read as "$n$ choose $k$"). These are also known as the "$k$-combinations" of $n$ elements. They satisfy the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- **Binomial Theorem.** For an integer $n \geq 0$, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.
- **Number of subsets.** There are $2^n$ distinct subsets of a set with $n$ elements.
- **Inclusion-Exclusion.** Suppose that $A_1, A_2, \ldots, A_n$ are sets. Then

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$
$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|,$$

and in general, with $A_I := \bigcap_{i \in I} A_i$,

$$|A_1 \cup \cdots \cup A_n| = \sum_{I \subseteq [1,n]} (-1)^{|I|} A_I.$$

Warm Up

1. (a) You and a friend each choose a number from 1 to 19 each day for a year. Show that there must be two identical days – i.e., you chose the same number on both days, and your friend also chose the same number on both days.

   (b) There are approximately 25,000 in-state students at LSU, and approximately 500 high schools in Louisiana. Show that there is a group of at least 50 LSU students who all went to same high school.

2. A Frozen Yogurt shop offers 7 flavors: Apricot, Blackberry, Cherry, Dragon Fruit, Elderberry, Fig, and Guava. How many distinct choices are there for each of the following options?

   (a) A *Large Cone* consists of any three scoops. Repeated flavors are allowed, and since the top scoops will drip onto the lower, the order matters!

   (b) A *Sundae* consists of any three scoops in a bowl. Repeated flavors are allowed, but the order doesn't matter.

   (c) A *Swirl Cone* allows the choice of any two distinct flavors.

   (d) A *Sampler Flight* is a sequence of three distinct flavors.

3. The total grade in a certain class is calculated by taking the average of the Exam and Homework grades. Out of 50 students, 30 earned an average grade of B or better on their Exams, and 35 earned an average grade of B or better on the Homework. What is the minimum number of students who earned a total grade of B or better?

---

Main Problems

---

4. (a) Determine the number of solutions to

$$x_1 + x_2 + x_3 = 13,$$

   where $x_1, x_2$, and $x_3$ are non-negative integers.

   (b) Determine the number of solutions to the above equation with $x_1 \geq 2$.

   (c) Finally, determine the number of solutions with the following constraints:

$$x_1 \geq 2, \quad x_2 \leq 3, \quad x_3 \leq 6.$$

   *Hint: First count the number of solutions with $x_2 > 3$ and no restriction on $x_3$, and then count the number of solutions with $x_3 > 6$ and no restriction on $x_2$. Now apply Inclusion-Exclusion.*

5. (a) A village has exactly $3^6 = 729$ voting residents, and the mayor is quite unpopular – in fact, only 70 residents are guaranteed to support him! The voting system is rather complicated: All voters are divided into equal groups, each group is itself divided into equal groups, and so on. At each level the groups must have the same number of people. In the smallest groups an elector is chosen by majority vote, these electors then choose the representatives of the larger groups they are members of, and so on. In the end, the representatives of the largest groups elect the mayor. (Note: If there is a tie within a group, then the opposition candidate wins.)

   Most importantly, the mayor himself gets to choose the groups for the upcoming election. Although his advisors have expressed alarm, the mayor assures them that he is not concerned about his low level of support. Why is the mayor so confident? How will he arrange the groups of voters?

(b) [Adapted from **Gelca-Andreescu 954**] It is now time for the presidential election in the (fictional!) country of Anchuria. With a population of 20 million voters, the election will be held in the same manner as described for the village above, so that the voters are divided into equal-size groups arranged in a tree-like structure. In this case, exactly 1% of voters support the current president. Is it possible for her to win re-election?

6. (a) [Adapted from VTRMC **1990 #8**] Ten points in space, no three of which are collinear, are connected, each one to all the others, by a total of 45 line segments. The resulting framework $F$ is "disconnected" into two disjoint nonempty parts if all 9 edges from a single vertex are removed. Prove that $F$ cannot be similarly disconnected by the removal of any 8 edges.

   (b) Part (a) shows that 9 edges must be cut in order to disconnect a single vertex. How many edges must be cut to disconnect a collection of $k$ vertices from the remaining $10 - k$? Determine the worst case – what number of vertices requires the **largest** number of cuts to disconnect?

7. The *Fibonacci sequence* is defined by $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

   (a) Calculate $F_2, F_3, \ldots, F_9$.

   (b) Prove that for $n \geq 1$,

   $$\sum_{k \geq 0} \binom{n-k}{k} = F_n.$$

   Note that $\binom{n}{k}$ is defined to be 0 if $n$ and $k$ are not non-negative integers with $k \leq n$. Check the first few values to be sure that you understand the notation!

   (c) Find (and prove) a simple formula for the expression

   $$F_n^2 - F_{n+1}F_{n-1}.$$

8. [Putnam **1957 B4**] Show that the number of ways of representing $n$ as an ordered sum of 1s and 2s equals the number of ways of representing $n+2$ as an ordered sum of integers greater than one. For example: $4 = 1+1+1+1 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2$ (5 ways) and $6 = 4+2 = 2+4 = 3+3 = 2+2+2$ (5 ways).

9. [Putnam **1980 A2**] Let $r$ and $s$ be positive integers. Derive a formula for the number of ordered quadruples $(a, b, c, d)$ of positive integers such that

   $$3^r \cdot 7^s = \mathrm{lcm}[a, b, c] = \mathrm{lcm}[a, b, c] = \mathrm{lcm}[a, c, d] = \mathrm{lcm}[b, c, d].$$

   The answer should be a function of $r$ and $s$.

   (Note that $\mathrm{lcm}[x, y, z]$ denotes the least common multiple of $x, y, z$.)