- Virginia Tech Mathematics Contest. Sat., Oct. 26. **Sign-up deadline: Sep. 27**.
- Putnam Mathematical Competition. Sat., Dec. 7. **Sign-up deadline: Nov. 1**.

LSU Problem Solving Seminar - Fall 2019
Sep. 18: Calculus

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Useful facts and strategies:

- **Intermediate Value Theorem.** Suppose that $f(x)$ is a continuous function defined on the interval $[a, b]$, and r is a value in between $f(a)$ and $f(b)$, so that

  $$f(a) < r < f(b) \quad \text{or} \quad f(a) > r > f(b).$$

  Then there is some point c in the interior of the interval, $a < c < b$, such that $f(c) = r$.

  In other words, a continuous function cannot "skip" any values.

- **Differentiability.** A function f is differentiable at a if the following limit exists:

  $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

  If so, this value is denoted by $f'(a)$.

- **Mean Value Theorem.** Suppose that $f(x)$ is differentiable on the interval $[a, b]$. Then there is a point $a < c < b$ such that

  $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  In other words, a differentiable function must achieve its **average** *slope at some point.*

- **Critical Points.** If $f(x)$ is a differentiable function on an interval $[a, b]$, then its maxima/minima must occur at the end points or the **critical points**, where are those x such that $f'(x) = 0$. The maxima/minima are classified by the negativity/positivity of $f''(x)$.

  Something similar is true for multivariable functions; the maxima/minima of $f(x, y)$ also occur when $\frac{\partial}{\partial x} f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are zero, but there is the additional possibility of a *saddle point.*

- **L'hospital's Rule.** Suppose $f(x)$ and $g(x)$ are differentiable. If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an **indeterminate form** (i.e., $\frac{0}{0}$ or $\frac{\infty}{\infty}$), then

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

- **Continuity and Limits.** If g is a continuous function (or even just has a limit that exists at $f(a)$), then

  $$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

- **Derivative Zero.** The only functions that satisfy $f'(x) = 0$ for all x are the constant functions $f(x) = C$.

- **Taylor Series.** The Taylor series of $f(x)$ around $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

  In general this series will hold for values of x in some interval around a.

- **Ordinary Differential Equations.** A differential equation of the shape

$$f'(x) = g(x) \cdot f(x)$$

  has a solution $f(x) = e^{\int g(x)dx}$, where $\int g(x)dx$ denotes an antiderivative of g.

Warm Up

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1. (a) Yesterday the High Temperature was 92 Degrees, and the Low Temperature was 72. Today the High Temperature was 93 Degrees, and the Low Temperature was 70. Show that at some time of day the temperature was identical on both days.

   Hint: Let $f(t)$ be the difference between Today's temperature at time t and Yesterday's temperature at time t, and use the Intermediate Value Theorem.

   (b) Now suppose that Tomorrow the High Temperature is 95 and Low Temperature is 73. Show (by sketching a plot of the temperatures) that it is possible for Tomorrow's temperature to be **different** from Today's at all times of day.

   (c) What is the strongest statement that can be concluded about the situation in part (b)? Must there be a time at which Today and Tomorrow's temperature differ by at most 2 degrees? By 1 degree?

2. Calculate the following limits.

   (a)

$$\lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{x}.$$

   Hint: It is not helpful to use L'hospital's Rule on the quotient above (why?). One alternative approach is to make the change of variables $x = \frac{1}{y}$, and then rearrange the expression.

   (b)

$$\lim_{x \to 0} \frac{\sqrt{1 - \cos(2x)}}{x}.$$

   Hint: Squaring is a **continuous** *operation, and thus* $(\lim f(x))^2 = \lim f(x)^2$.

3. The *Logarithmic Derivative* of a function $f(x)$ is defined to be $\frac{f'(x)}{f(x)}$.

   (a) Explain the name by using the Chain Rule to calculate $\frac{d}{dx}\log(f(x))$.

   (b) Using properties of logarithms, this can be used to simplify the use of the Product Rule for multiple terms. Calculate the derivative of $f(x) = x(x+1)(x+2)(x+3)(x+4)$.

   (c) It can also be used to simplify Differential Equations. Find the solution to $y' = 2y$ with initial condition $y(0) = 3$.

Main Problems

4. Determine $\lim_{x\to 0} \dfrac{x^{2019} - (\sin x)^{2019}}{x^{1010}(\sin x)^{1011}}$.

5. A Sno-Cone company serves its product in paper cones that are in the shape of right cylindrical cones, and have a volume of 1 Liter. In order to save costs, they would like to determine the dimensions of the cones that require the minimum amount of material – i.e., the smallest possible surface area. Determine the optimal dimensions of the cones.

   You may use without proof the fact that if the base radius is r, and the height is h, then the volume is $V = \frac{1}{3}\pi r^2 h$, and the surface area is $A = \pi r\sqrt{r^2+h^2}$ (though you should also try to derive these formulas!).

6. (a) Find a continuous function that takes every value in its range an odd number of times.

   Hint: 1 is an odd number.

   (b) Find a continuous function that takes every value in its range more than once (i.e., fails the Horizontal Line Test), and takes each value an odd number of times.

   (c) Is it possible for a continuous real function to take every value in its range **exactly** twice?

7. [**Gelca-Andreescu 470**] Does there exist a continuous function $f : [0,1] \to \mathbb{R}$ that assumes every element of its range an even (and finite) number of times?

   Hint: If you answered Problem 6 (c) correctly, then you know that such an f cannot assume every value twice...

8. (a) Solve the differential equation $y' = xy$ with initial condition $y(1) = 2$.

   (b) Verify that the function $y(x) = e^{e^x - 1}$ satisfies the differential equation $y' = e^x y$, with initial condition $y(0) = 1$.

9. [VTRMC **1996 #3**] Solve the differential equation $y^y = e^{dy/dx}$ with the initial condition $y = e$ when $x = 1$.

10. [Putnam **2014 A1**] Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

   about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.