- Virginia Tech Mathematics Contest. Sat., Oct. 26. **Sign-up deadline: Sep. 27**.
- Putnam Mathematical Competition. Sat., Dec. 7. **Sign-up deadline: Nov. 1**.

LSU Problem Solving Seminar - Fall 2019
Sep. 25: Number Theory

Prof. Karl Mahlburg
Website: `www.math.lsu.edu/~mahlburg/teaching/Putnam.html`

Useful facts and strategies:

- **Divisibility Tests.** A positive integer n is divisible by:
  - 2 if its last digit is a multiple of 2;
  - 3 if the sum of its digits is a multiple of 3;
  - 4 if its last two digits are a multiple of 4;
  - 5 if its last digit is 0 or 5;
  - 7 if ______________________________
  - 9 if the sum of its digits is a multiple of 9;
  - 11 if the alternating (with plus and minus signs) sum of its digits is a multiple of 11.
- The current year has the prime factorization $2019 = 3 \cdot 673$, and the previous year $2018 = 2 \cdot 1009$.
- **Fermat's Little Theorem.** If p is prime and a is any integer, then $a^p - a$ is a multiple of p.
- **Remainders of Squares.** For any integer n, n^2 can only have the following remainders:

  0 or 1 when divided by 3; 0 or 1 when divided by 4; 0, 1, 4, 5, 6, or 9 when divided by 10.

- **Greatest Common Divisors / Least Common Multiples.** If a and b are integers with prime factorizations $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, b = p_1^{\beta_1} \cdots p_r^{\beta_r}$, their greatest common divisor is

  $$\gcd(a,b) = p_1^{\min\{\alpha_1,\beta_1\}} \cdots p_r^{\min\{\alpha_r,\beta_r\}}.$$

  The least common multiple is found by replacing min by max.

  The equation $ax + by = N$ has integer solutions if and only if $\gcd(a,b)$ divides N.

- **Euler's Totient Function.** If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then define

  $$\phi(n) := n \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

  Among the integers $1, 2, \dots, n$, exactly $\phi(n)$ of them satisfy $\gcd(a,n) = 1$. Furthermore, if $\gcd(a,n) = 1$, then $a^{\phi(n)}$ has remainder 1 when divided by n.

- **Casting Out Nines.** If n is an integer and s is the sum of its decimal digits, then $n - s$ is a multiple of 9.

Warm Up

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1. Answer these without using a calculator!
   (a) Is 12026 a multiple of 6? Of 7?
   (b) Find the prime factorization of the following numbers:
      (i) 407;
      (ii) 707;
      (iii) 1007.
2. You wrote down a friend's phone number one night, but when you checked the next day, two of the digits were smudged out:

$$(2\,3\,\underline{\;?\;}\,)\;\underline{\;?\;}\,1\,9\;-\;2\,5\,2\,8.$$

   Fortunately, your friend (being a mathematician) excitedly pointed out that the number is a multiple of 99 when read as a single 10-digit number. Determine the two missing digits.
3. One of the following two expressions is a multiple of 19 – determine which one:

$$7^8 + 8^7 \qquad \text{or} \qquad 7^7 + 8^8?$$

Main Problems

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4. What are the last two digits of 2019^{2019}?
5. [**Gelca-Andreescu 845**] Find the integers n for which $\dfrac{n^3 - 3n^2 + 4}{2n-1}$ is an integer.
   Hint: It is easier to reduce the fraction if you first multiply the numerator by 8*. And since the denominator is odd, this has no effect on whether or not the expression is an integer (why?).*
6. (a) Suppose that n and b are positive integers, and a is a (positive) divisor of b. Prove that $n^a - 1$ is a divisor of $n^b - 1$.
   (b) Find the prime factorization of $2^{16} - 1 = 65535$.
7. (a) [VTRMC **1981 #1**] The number $2^{48} - 1$ is exactly divisible by what two numbers between 60 and 70?
   (b) Is $2^{48} - 1$ a multiple of 2019?
8. A real number is *rational* if it is the quotient of two integers, i.e. $\frac{r}{s}$ with $s \neq 0$.
   (a) Prove that $\sqrt{2}$ is irrational. The standard approach is to suppose to the contrary that $\sqrt{2} = \frac{r}{s}$ for integers r and s, so that

$$2 = \frac{r^2}{s^2} \implies 2s^2 = r^2.$$

   Now describe the powers of 2 dividing both sides to reach a contradiction.

(b) Are there any integers n such that $\sqrt{n}$ is rational?

(c) Are there any linear combinations of the form $a + b\sqrt{2}$ that are rational, where a is an integer b is a nonzero integer?

9. [Putnam **1955 A1**] Prove that if a, b, c are integers and $a\sqrt{2} + b\sqrt{3} + c = 0$, then $a = b = c = 0$.

10. (a) Find the fraction with the smallest denominator that lies strictly between $\frac{1}{2020}$ and $\frac{1}{2019}$.

    (b) Given two nonnegative rational numbers $\frac{p}{q} < \frac{r}{s}$, the *mediant* is defined to be $\frac{p+r}{q+s}$. Show that the mediant always lies between the two original fractions, so that

$$\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}.$$

    (c) Find an example of two fractions $\frac{p}{q} < \frac{r}{s}$ such that $\frac{c}{d}$ lies between them, and d is smaller than either q or s.

11. [Putnam **1993 B1**] What is the smallest integer $n > 0$ such that for any integer m in the range $1, 2, 3, \ldots, 1992$ we can always find an integral multiple of $\frac{1}{n}$ in the open interval $\left(\frac{m}{1993}, \frac{m+1}{1994}\right)$?