- Virginia Tech Mathematics Contest. Sat., Oct. 26. **Sign-up deadline: Sep. 27**.
- Putnam Mathematical Competition. Sat., Dec. 7. **Sign-up deadline: Nov. 1**.

LSU Problem Solving Seminar - Fall 2019

Oct. 2: Geometry and Trigonometry

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Useful facts and strategies:

- **Triangle Inequality.** If a, b, c are the side lengths of a triangle, then $a < b + c$.
- **Pythagorean Theorem.** Suppose that ABC is a right triangle, with $\angle ABC = 90°$. If the (opposing) side lengths are $|\overline{AB}| = c$, $|\overline{AC}| = b$, $|\overline{BC}| = a$, then $b^2 = a^2 + c^2$.
- **Law of Cosines.** If a triangle has sides of lengths a, b, and c, and α is the angle opposite the side of length a, then
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$
- **Law of Sines.** If β is the angle opposite b, and γ is the angle opposite c, then
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} = \frac{1}{2R},$$
where R is the radius of the circumscribed circle (which contains the vertices of the triangle).
- **Pythagorean Formula.** For all x,
$$\sin^2(x) + \cos^2(x) = 1,$$
- **Addition Formulas.** For all x and y,
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$
- **Heron's Formula.** If a triangle has side lengths a, b, and c, then its area is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s := \frac{a+b+c}{2}$ is the *semiperimeter.*
- **Prisms.** The volume of a prism of height h and base area A is $V = \frac{hA}{3}$.
- **Dot Products and Projection.** For two vectors $\vec{v_1} = (x_1, y_1, z_1), \vec{v_2} = (x_2, y_2, z_2)$, their dot product is
$$\vec{v_1} \cdot \vec{v_2} = x_1x_2 + y_1y_2 + z_1z_2 = ||v_1|| \cdot ||v_2|| \cos(\theta),$$
where θ is the angle between the two vectors.
- **Cross Products and Area.** In three dimensions,
$$\vec{v_1} \times \vec{v_2} = (y_1z_2 - y_2z_1)\mathbf{i} + (x_2z_1 - x_1z_2)\mathbf{j} + (x_1z_2 - x_2z_1)\mathbf{k},$$
and the parallelogram spanned by $\vec{v_1}, \vec{v_2}$ has area $||\vec{v_1} \times \vec{v_2}||$. In two dimensions, the area of the parallelogram spanned by $(x_1, y_1), (x_2, y_2)$ is $|x_1y_2 - x_2y_1|$.

Warm Up

1. Suppose that an equilateral triangle is *inscribed* in a circle (this means that the vertices of the triangle are all **on** the circle); we also say that the circle is *circumscribed* around the triangle. For simplicity, suppose that the circle has radius 1.

   (a) What is the area of the triangle?

   (b) Is the triangle more or less than half the area of the circle? Try to answer this without using a calculator!

2. Suppose that two right triangles of heights h_1 and h_2 share the same base, but are oppositely oriented, as in Figure 1. Your goal is to determine h, the height of the point of intersection.

   (a) Find h if the triangles are congruent ($h_1 = h_2$).

   (b) Find a general formula for h in terms of h_1 and h_2.

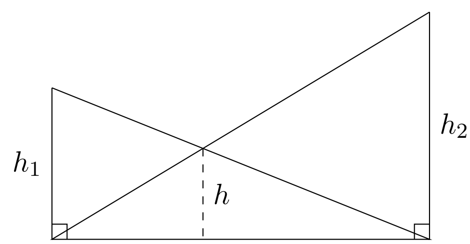

Figure 1: Triangles for Problem 2.

3. Prove the double-angle formula for the cosecant function:

$$\csc(2\theta) = \frac{\cot\theta + \tan\theta}{2}.$$

   One approach is to draw a right triangle with $\angle ABC = 90°$, and $\angle BAC = \theta$. Now draw a perpendicular line segment from B to AC, and let this have length 1. You should now be able to label line segments with lengths $\cot\theta, \tan\theta, \sec\theta$, and $\csc\theta$. Calculate the area of the triangle in two different ways.

   Remark: There are several other approaches: 1. Draw the triangle ABC as above, but then draw a congruent copy reflected through AB to directly obtain an angle of 2θ. 2. Use Euler's identity $e^{ix} = \cos x + i \sin x$. 3. Use half-angle formulas for cotangent and tangent.

Main Problems

4. A *chord* in a circle is a line segment that joins two points of the boundary. The *angle* of a chord is the angle formed by drawing radii from the center of the circle to the two endpoints of the chord.

(a) Find the relationship between the chord length ℓ and its angle θ (in other words, try to express ℓ in terms of θ and vice versa).

(b) Determine the area cut out by a chord with angle θ. This is the area of the region on the "outside" of the chord – that does **not** contain the center of the circle.

Remark: As a quick check for parts (a) and (b), compare to your answers for Problem 1.

(c) Suppose that two circles intersect at right angles (in other words, if radii are drawn to the intersection points, they form a right angle). See Figure 2. If the circles have radius 1 and r, determine the total area covered by the overlapping circles.

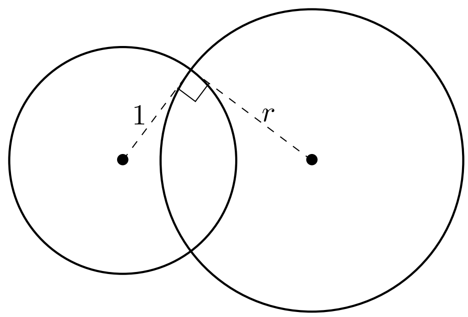

Figure 2: Overlapping circles for Problem 4.

5. (a) If a circle of radius r is inscribed in a square, what is the area of the square?

   (b) [VTRMC **1988 #1**] A circle C of radius r is circumscribed by a parallelogram S. Let θ be one of the interior angles of S, with $0 < \theta \leq \frac{\pi}{2}$. Calculate the area of S as a function of r and θ.

6. [**Gelca-Andreescu 695**] Let ABC be a triangle, with D and E on the respective sides AC and AB. If M and N are the midpoints of BD and CE, prove that the area of the quadrilateral $BCDE$ is four times the area of the triangle AMN.

7. (a) Find a circle that passes through the points $(0, 0), (2, 0)$, and $(0, 4)$.

   (b) Prove that given any three non-collinear points in the plane, there is a unique circle that passes through all of them.

   Hint: There are two main approaches: 1. Euclidean Geometry – Consider the midpoints of the triangle formed by the three points. 2. Analytic Geometry – write down the equation of a general circle, and plug in the three points.

8. [Putnam **1961 B3**] Suppose that four points in the plane are such that no three are collinear, and all four do not lie on a circle. Prove that one point lies inside the circle through the other three.

9. [Putnam **1977 B4**] Let C be a continuous closed curve in the plane which does not cross itself and let Q be a point inside C. Show that there exist points P_1 and P_2 on C such that Q is the midpoint of the line segment P_1P_2.