- Virginia Tech Mathematics Contest. Sat., Oct. 26.
- Putnam Mathematical Competition. Sat., Dec. 7. **Sign-up deadline: Nov. 1**.

LSU Problem Solving Seminar - Fall 2019
Oct. 9: Series and Recurrences

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Useful facts and strategies:

- **Terminology.** A *sequence* is an ordered list of numbers: $\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \dots$.
A *series* is the sum of a sequence: $\sum_{n=1}^{\infty} a_n$.
- **Limit of a Sequence.** A sequence $\{a_n\}_{n=1}^{\infty}$ *converges to a limit* ℓ if for any $\varepsilon > 0$ there is an N such that $|a_n - \ell| < \varepsilon$ for all $n > N$.
- **Geometric Series.** If $|x| < 1$, then $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$.
- **Ratio Test.** Let $L := \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$. If $L < 1$, then $\sum_{n\geq 1} a_n$ converges, and if $L > 1$, then the sum diverges. If $L = 1$, the test is inconclusive.
- **Monotone Convergence.** If $a_1 \leq a_2 \leq \dots$ and all $a_n \leq B$ for some constant B, then $\lim_{n\to\infty} a_n$ exists (though it may be less than B).
- **Alternating Series.** If $a_1 \geq a_2 \geq \dots$ and $\lim_{n\to\infty} a_n = 0$, then the alternating series $a_1 - a_2 + a_3 - a_4 + \dots$ converges.
- **Integral Comparison.** If $f(x)$ is a decreasing function for $x \geq 0$, then $\sum_{n\geq 1} f(n) \leq \int_0^{\infty} f(x)dx$.
- **Linear Recurrences.** The *characteristic polynomial* associated to a (homogeneous) recurrence $a_{n+k} = c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n$ is $p(x) := x^k - c_{k-1}x^{k-1} - \cdots - c_1 x - c_0$. If $p(x)$ has **distinct** roots $\lambda_1, \dots, \lambda_k$, then the general solution to the recurrence is

$$a_n = b_1\lambda_1^n + \cdots + b_k\lambda_k^n,$$

where the constants are determined by k initial values. If there is a **repeated** root λ of order m, then the general solution has the term $(d_{k-1}n^{k-1} + \cdots + d_1 n + d_0)\lambda^n$.

Warm Up

1. Evaluate the following sums:

   (a) $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots$.

   (b) $1 + \dfrac{2}{3} + \dfrac{4}{9} + \dfrac{8}{27} + \cdots$.

2. Solve the following recurrences; this means finding a closed form expression for a_n. For example, the recurrence with $a_1 = 1$ and $a_{n+1} = a_n + 1$ has solution $a_n = n$.

   (a) $a_0 = 0$, and $a_{n+1} = a_n + 2n + 1$ for $n \geq 1$.

   (b) $a_0 = 1$, and $a_{n+1} = 2a_n$ for $n \geq 1$.

   (c) $a_0 = -1$, and $a_{n+1} = 2a_n + 2n + 1$ for $n \geq 1$.
   Hint: Begin by finding the "non-homogeneous solution" by letting $a'_n = dn+e$, and choosing the constants such that $a'_{n+1} = 2a'_n + 2n + 1$. Then let $a_n = a'_n + b_n$, and get a simpler a recurrence for b_n.

Main Problems

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3. (a) Evaluate the finite series
$$1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^N}.$$
   Try to do this inductively: calculate the first few values so that you can guess a formula, and then prove it!

   (b) Prove the general summation formula for the finite geometric series:
$$S_N(x) := 1 + x + x^2 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}.$$
   Do this by considering $S_N(x) - xS_N(x)$.

   (c) Now evaluate
$$1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \cdots + \frac{N+1}{2^N}.$$
   Again, try to do this inductively: calculate the first few values, and write them in the form $4 - \frac{a_N}{2^N}$. For example, with $N = 2$, the sum is $\frac{11}{4} = 4 - \frac{5}{4}$.

   (d) Find the general summation formula for
$$T_N(x) := 1 + 2x + 3x^3 + \cdots + (N+1)x^N.$$
   Do this by considering $T_N(x) - xT_N(x)$, and recalling part (b).

4. [VTRMC **1990 #6**] The number of individuals in a certain population (in arbitrary real units) obeys, at discrete time intervals, the equation
$$y_{n+1} = y_n(2 - y_n) \quad \text{for } n = 0, 1, 2, \ldots,$$
   where y_0 is the initial population.

   (a) Find all "steady-state" solutions y^* such that if $y_0 = y^*$, then $y_n = y^*$ for $n = 1, 2, \ldots$.

   (b) Prove that if y_0 is any number in $(0, 1)$, then the sequence $\{y_n\}$ converges monotonically to one of the steady-state solutions found in (a).

5. (a) Evaluate $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$.

   Properly, you should interpret this as the limit of the sequence defined by $a_0 = \sqrt{6}$, and $a_{n+1} = \sqrt{6+a_n}$ for $n \geq 0$.

   (b) Evaluate $\sqrt{6-\sqrt{6-\sqrt{6-\cdots}}}$.

6. [**Gelca-Andreescu 409**] Find the positive real solutions to the equation

$$\sqrt{x+2\sqrt{x+\cdots+2\sqrt{x+2\sqrt{3x}}}} = x.$$

7. A sequence is defined by $a_1 = 1, a_2 = 1, a_3 = 4$, and

$$a_n = \frac{a_{n-1}a_{n-2}-1}{a_{n-3}} \quad \text{for } n \geq 4.$$

   (a) Calculate the next several values of a_n. Note that the definition of the recurrence includes division by a_{n-3}. Do any of the values that you calculated have denominators?

   (b) Use your data to calculate $a_n + a_{n-4}$. You should notice a striking relationship to some other terms in the sequence. Prove your observation.

   (c) Finally, use this to to further prove that every a_n is an integer.

8. (a) Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence. Show that

$$\sum_{m\geq 1}\sum_{n\geq 1}\frac{2}{a_m(a_m+a_n)} = \sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{a_m a_n}.$$

   Hint: What happens if you interchange m and n?

   (b) Evaluate $\displaystyle\sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{2^m(2^m+2^n)}$.

9. [Putnam **1999 A4**] Sum the series

$$\sum_{m\geq 1}\sum_{n\geq 1}\frac{m^2 n}{3^m(n3^m+m3^n)}.$$