- Virginia Tech Mathematics Contest. Sun., Oct. 27.
- Putnam Mathematical Competition. Sat., Dec. 7. **Sign-up deadline: Nov. 1**.

LSU Problem Solving Seminar - Fall 2019
Oct. 16: Probability

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Useful facts and strategies:

- **Probability Spaces.** A (countable) probability space consists of a set of distinct events $A_1, A_2, \ldots$ and a probability function $0 \leq p \leq 1$ such that $p(A_1) + p(A_2) + \cdots = 1$.

  In a finite probability space, typically $p(A) = \dfrac{\#\text{ outcomes in } A}{\#\text{ total outcomes}}$. For example, if two dice are rolled, there are 5 ways to obtain a sum of 6 (namely, $5+1, 4+2, 3+3, 2+4, 1+5$), and $6^2 = 36$ total combinations, so $p(6) = \frac{5}{36}$.
- **Random Variables and Expectation.** A random variable X assigns a real value x to each event A. The expected value, or *average* of X is

  $$E[X] := \sum_x x \cdot P(X = x).$$

  For example, the expected number of Tails when two coins are flipped is $0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4} = 1$.
- **Additivity of Expectation.** If X and Y are random variables, $E[X+Y] = E[X] + E[Y]$.
- **Exponential and Stirling approximation.** Use the following formulas to approximate discrete probabilities for large n (and small k):

  $$\left(1 - \tfrac{1}{n}\right)^n \sim e^{-1}, \qquad n! \sim \left(\tfrac{n}{e}\right)^n, \qquad \text{and} \qquad \binom{n}{k} \sim \frac{n^k}{k!}.$$

Warm Up

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1. A 3-set tennis match ends whenever either player has won 2 sets, which requires a total of either 2 or 3 sets to be played.
   (a) Suppose that each player is equally likely to win each set. Show that the probability that the match ends in 2 sets is the same as the probability that it ends in 3 sets.
   (b) Now suppose that the players are not evenly matched, and that each set is independent. In other words, Player 1 has probability p of winning any set. Is it more likely that the match ends in 2 sets, or 3 sets?
2. (a) How many digits are there in 100^{100}?
   (b) How many digits are there in 99^{99}?
   Hint: Why is this a probability question?

3. Suppose that 5 dice are rolled.
   (a) What is the probability that the sum is a multiple of 6?
   (b) What is the probability that the sum is greater than or equal to 18?
   (c) What is the probability that the product is a multiple of 6?

Main Problems

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4. A classroom floor is covered with square tiles that measure 12 inches per side.
   (a) If a medium (12-inch diameter) pizza is dropped on the floor, what is the probability that it does **not** cover the corner of any tile?
   (b) Now suppose that a square pizza that measures 12 inches per side is dropped on the floor. What is the probability that it does not cover any corner?
   Hint: In this case, you will need to consider all possible angular rotations of the pizza. . . .

5. (*Non-transitive dice*) Consider a set of three six-sided dice, where the sides are labeled by the following values:

$$A : 2, 2, 6, 6, 7, 7;$$
$$B : 1, 1, 5, 5, 9, 9;$$
$$C : 3, 3, 4, 4, 8, 8.$$

   Note that each die has an average side value of 5.
   (a) If both A and B are rolled, which die is more likely to "win" (with a higher value)?
   (b) Which die is more likely to win between B and C?
   (c) Based on your answers above, which die do you think is more likely to win between A and C? Now calculate the actual probability.

6. (a) Suppose that Player 1 flips 1 coin, and Player 2 flips 2 coins. Player 2 wins if she has **strictly** more Heads than Player 1, and otherwise Player 1 wins. Would you rather be Player 1 or 2?
   (b) Now suppose that Player 1 flips 11 coins, and Player 2 flips 12 coins; again, Player 2 wins if she has more Heads than Player 1. Would you rather be Player 1 or 2?

7. (a) A chemistry lab has a large supply of glass test tubes that are 10 inches long. One end of the tubes has a small opening, which is colored Red. One day a clumsy assistant drops a box full of tubes, all of which break into 2 pieces. Assuming that the breakage point is random, what is the average length of the pieces with the Red marking?
   (b) The lab receives an order of replacement test tubes, which are even thinner (and thus more fragile) than before. After a few months, an entire box of the new test tubes are dropped, and they all break into 3 pieces. Assuming that the breakage points are distributed evenly, what is the average length of the pieces with the Red marking?

8. (a) [**Gelca-Andreescu 1085**] Two people play the coin tossing game of "War": each player flips a coin, and if they match, then Player 1 gets both coins, but if they differ, Player 2 gets both coins. If Player 1 begins with m coins, and Player 2 begins with n coins, what is the expected length of the game?

   Hint: Although it may appear that there are two variables (the number of coins held by Players 1 and 2), note that the total number of coins is constant throughout the game.

   (b) What is the probability that Player 1 wins?

9. [Putnam **2001 A2**] You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.