LSU Problem Solving Seminar - Fall 2019
Oct. 23: Invariance and Games

Thanks to Farid Bouya *for preparing this week's problem sheet!*

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- **Virginia Tech Mathematics Contest:** Sun., Oct. 27, 8:30 – 11:30, Lockett 232.
- **Format.** This written exam consists of 7 problems to be solved in 2 hours and 30 minutes.
- **Grading.** Each problem is graded out of **10** points, for a maximum possible score of **70**. The grading is strict, with little partial credit given for guesses or incomplete proofs.
- **Unordered.** The problems are **not** ordered by difficulty, so you should plan on spending the first 15–20 minutes reading all of the problems and then deciding which ones you are best able to answer.
- **1 hour per write-up.** In order to get full credit, your solutions must be written very carefully. If you use a result from a course, refer to it by name (e.g. Fundamental Theorem of Calculus). After you solve a problem, you should plan on spending up to one hour writing your solution. Remember, it is better to solve one problem completely than several problems partially.

Useful facts and strategies:

- **Well-Ordering Property and Infinite Descent.** Every subset of natural numbers has a least element, or, equivalently, there is **no** infinite sequence of decreasing positive integers $n_1 > n_2 > n_3 > \cdots > 0$.

  This principle frequently applies to rational numbers as well; if r_i are rational numbers whose denominators are all **bounded**, then there is no infinite sequence $r_1 > r_2 > \cdots > 0$.
- **Invariants/Monovariants.** If you are asked about the possible outcomes of a procedure, try to find some *invariant* property that remains the same at each step. To show that a certain procedure ends in a **finite** number of steps, find a *monovariant:* a measurable quantity that is constantly increasing or decreasing.
- **Distance Functions.** When dealing with an n-tuple $S_k = (x_1, x_2, \dots, x_k)$ that gets manipulated in each step, it is sometimes useful to calculate its distance from the origin (in k-dimensional space): $d = \sqrt{\sum_{i=1}^n x_i^2}$ and $d^2 = \sum_{i=1}^n x_i^2$. If the distance is strictly increasing or decreasing, this can constrain the number of steps involved.

Warm Up

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1. A single-elimination knockout tournament takes place with n teams, with a bye used whenever the number of teams in the round is odd. How many games need to be played to determine the champion?

2. Start with positive integers $1, 2, \ldots, 4n$. In each move we choose 2 numbers a and b in the list, remove them, and insert $|a-b|$ instead. Show that the last number remaining is even.

3. Let $A_{m\times n}$ be a matrix with integer entries. In each step you can change the sign of all the entries in a row or a column. Show that it is possible to reach a state where A has nonnegative row and column sums.

   Hint: Consider the **sum** *of the entries in the matrix... what is the effect of changing the sign of a row or column sum that is negative?*

Main Problems

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4. Three boxes A, B, and C initially contain a, b, and c balls respectively.

   (a) At each move, one box will be chosen (say A), and the other two boxes (B and C) have one ball moved from each to A, so the total number of balls is unchanged. When (depending on a, b, and c) can we move all balls to one box?

   Hint: Consider some small examples: if $(a, b, c) = (2, 1, 4)$, it is possible, but if $(a, b, c) = (1, 2, 3)$ it is impossible.

   (b) At each move, one box will be chosen (say A), that receives a ball, and the other two boxes (B and C) lose a ball, so the total number of balls is reduced by 1. When (depending on a, b, and c) can we be left with a single ball?

   Hint: The changes in a, b, and c alone are not very helpful. What if we study their differences?

5. In each move we can choose two neighboring numbers and add the same integer to them. Can we get from the left setup to the right one?

1	2	3
4	5	6
7	8	9

7	8	9
6	2	4
3	5	1

   Hint: Try to do a suitable black and white coloring of the nine squares. Then consider the sum of black squares versus the sum of the white squares.

6. Consider the figure below. In each move we can switch the sign of every entry in one of the following:

   - A row
   - A column
   - Any of the two diagonals
   - Any skew line parallel to the diagonals

   For example, we can change the sign of any of the four entries in the corner.
   Can we get rid of the -1?

1	-1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

Hint: Find a suitable black and white coloring of the squares.

7. Consider ordered pairs of integers (x, y). At each step you are allowed to replace it by any of the following:

   - $(x + y, y)$,
   - $(x - y, y)$,
   - (y, x).

   (a) Show that if you start with $(1, 1)$, it is possible to reach $(19, 79)$.

   (b) If you start with $(8, 28)$, is it possible to reach $(18, 78)$?

   (c) In general, if we start with (a, b), which pairs can be reached?
   Hint: Does this remind you of the Euclidean algorithm?

8. (a) Start with a sequence $S_0 = (a_0, b_0, c_0, d_0)$ of integers and define

   $$S_{n+1} = (a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) := (|a_n - b_n|, |b_n - c_n|, |c_n - d_n|, |d_n - a_n|)\,.$$

   What is the long-run behavior of the sequence? Does it ever repeat or become fixed?

   (b) Start with a sequence $S_0 = (a_0, b_0, c_0, d_0)$ of distinct integers and define

   $$S_{n+1} = (a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = (a_n - b_n, b_n - c_n, c_n - d_n, d_n - a_n).$$

   Show that at least one entry becomes arbitrarily large.
   Hint: Note that $a_n + b_n + c_n + d_n = 0$ for $n \geq 1$. Now consider the distance from the origin, $a_n^2 + b_n^2 + c_n^2 + d_n^2 \ldots$.

9. Start with a sequence $S_0 = (x_1, x_2, \ldots, x_n)$ of pairwise distinct integers and define

   $$T(S) := \left(\frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \ldots, \frac{x_n + x_1}{2} \right).$$

   Show that $T, T^2, \ldots$ eventually leads to nonintegral components.

   Hint: Consider the average of S_0, namely $X := \dfrac{x_1 + \cdots + x_n}{n}$.

10. [IMO **1986 A3**] To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned numbers x, y, and z respectively and $y < 0$, then the following operation is allowed: The numbers x, y, and z are replaced by $x+y$, $-y$, and $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure comes to an end after a finite number of steps.

    Hint: The sum of squares is messy this time. What about sum of squares of differences? Try sum of squares of differences of vertices of distance 2, that is $(x - z)^2 + \cdots$.

11. [Putnam **1995 B5**] A game starts with four heaps of beans, containing 3, 4, 5, and 6 beans. The two players move alternately. A move consists of taking **either**

    (a) One bean from a heap, provided at least two beans are left behind in that heap, **or**

    (b) a complete heap of two or three beans.

    The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.