MATH 7230 Homework 4 - Fall 2019

Due Tuesday, Oct. 15 at 10:30

`https://www.math.lsu.edu/~mahlburg/teaching/2019-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

In Problems 1 – 2 you will give a proof of Childress Exercise 2.15, and explore some additional aspects of Dedekind zeta functions.

1. It is a basic fact from the theory of infinite products that if $|x_n| < 1$ and $\sum_{n\geq 1} x_n$ converges absolutely, then $\prod_{n\geq 1}(1+x_n)$ converges. If this result is unfamiliar, you may also submit Exercises 4 and 5 from
`https://www.math.lsu.edu/~mahlburg/teaching/handouts/2017-7230/HW2.pdf`.

   (a) Use the Euler product for the Dedekind zeta function to show that $\zeta_K(s)$ converges for $\mathrm{Re}(s) > 1$ if the following sum converges:
   $$\sum_{P\subset\mathcal{O}_K \text{ prime}} \frac{1}{N(P)^s}.$$

   (b) Now show that for real s, the above sum is bounded by
   $$[K:\mathbb{Q}]\cdot \sum_{p \text{ prime}} \frac{1}{p^s},$$
   and reach the desired conclusion.

2. As observed before Exercise 2.15,
$$\zeta_K(s) = \sum_{n\geq 1} \frac{\gamma_n}{n^s}, \qquad \text{where } \gamma_n := \#\left\{I\subset \mathcal{O}_K \mid N(I) = n\right\}.$$

   (a) Prove that if $K = \mathbb{Q}(\sqrt{D})$ is a quadratic field, then the γ_n are unbounded.

   (b) Prove that if $K = \mathbb{Q}(\zeta_m)$, then the γ_n are unbounded.
   Remark: In fact, the γ_n are always unbounded for any number field K, but showing this requires knowing that there are infinitely many primes that split – which follows from Theorem 2.5.1.

3. In this problem you will give a more elementary proof of Lemma 4.3 in Childress (the proof in the book uses the Galois tower for $\mathbb{Q}(\zeta_m)$). The statement is that if $(g, m) = 1$, then
$$\prod_{\chi\in\widehat{G}_m}(1-\chi(g)T) = \left(1-T^f\right)^{\frac{\phi(m)}{f}},$$
where f is the order of g in $(\mathbb{Z}/m\mathbb{Z})^\times$.

(a) Although this is not essential to the argument, it is a useful observation to begin by proving that if $m = p_1^{a_1} \cdots p_r^{a_r}$, then without loss of generality we can assume that $g \not\equiv 1 \pmod{p_j^{a_j}}$.

(b) Let $H := \langle g \rangle < (\mathbb{Z}/m\mathbb{Z})^\times$. Explain why there is a character $\chi_f \in \widehat{G}_m$ that satisfies $\chi_f(g) = \zeta_f$.

In particular, suppose that g' generates the maximal cyclic extension of H, so that

$$(\mathbb{Z}/m\mathbb{Z})^\times \cong \left\langle g' \right\rangle \times G^\perp,$$

where $g \in \langle g' \rangle$. Then if g' has order f', set $\chi_f(g') := \zeta_{f'}$ and $\chi_f(G^\perp) := 1$.

(c) Show that

$$\prod_{j=0}^{f-1} \left(1 - \chi_f^j(g)T\right) = 1 - T^f.$$

Then let $\widehat{H} := \langle \chi_f \rangle < \widehat{G}_m$, and show that

$$\prod_{\chi \in \widehat{H}} (1 - \chi(g)T) = \left(1 - T^f\right)^{\frac{f'}{f}}.$$

(d) Finally, let $\widehat{S}_H$ be a set of coset representatives of $\widehat{H}$, so that $\widehat{G}_m = \bigsqcup_{\psi \in \widehat{S}_H} \psi \widehat{H}$.

Prove that for any $\psi \in \widehat{S}_H$,

$$\prod_{\chi \in \psi \widehat{H}} (1 - \chi(g)T) = \left(1 - T^f\right)^{\frac{f'}{f}}.$$

Hint: Show that $\psi(g)$ is an f-th root of unity.

4. In this problem you will answer Childress Exercise 2.16. Bertrand's postulate states that for any $n \geq 1$, there is a prime $n < p \leq 2n$. There is a famous elementary proof due to Erdös that uses binomial coefficients.

   Use this fact to construct an infinite set of primes with Dirichlet density zero.