MATH 7230 Homework 5 - Fall 2019

Due Tuesday, Oct. 29 at 10:30

`https://www.math.lsu.edu/~mahlburg/teaching/2019-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

Problems 1 – 2 explore explicit calculations in ideal and ray class groups, as in Step 1 from the proof of Childress Proposition 2.1.

1. Recall that ideal class groups can be calculated using Minkowski's Bound (Ash 5.3.5), which states that if $F = \mathbb{Q}(\sqrt{-D})$, then any ideal class contains an I such that $N(I) \leq \frac{2}{\pi}\sqrt{\mathrm{Disc(F)}}$.

   (a) Suppose that $D = 5$. Verify that
   $$\mathcal{CL}(F) \cong C_2 = \{(1), P_2\},$$
   where $P_2 = (2, 1+\sqrt{-5})$ (which accounts for the ramification $(2) = P_2^2$).

   (b) We also discussed in lecture that $(1+\sqrt{-5}) = P_2P_3$, with $P_3 = (3, 1+\sqrt{-5})$. However, if one wanted to work with the ray at (6), then P_2 and P_3 must be excluded. Show that one can alternatively represent the class group as
   $$\mathcal{CL}(F) \cong C_2 = \{(1), P_7\},$$
   where $P_7 = (7, 3+\sqrt{-5})$. In particular, show that $(3+\sqrt{-5}) = P_2P_7$.

2. Now you will show that the class group for $F = \mathbb{Q}(\sqrt{-21})$ is isomorphic to $C_2 \times C_2$.

   (a) Apply Minkowski's bound, and determine the factorization of $(2), (3)$, and (5).

   (b) Now you must search for any (multiplicative) relations among these prime ideals, relative to the quotient by the principal fractional ideals. First, show that P_2 and P_3 are independent: assume to the contrary that $P_2 = (\alpha)P_3$ for some $\alpha \in \mathbb{Z}[\sqrt{-21}]$ and reach a contradiction.

   (c) Next, show that P_5 is generated by P_2 and P_3 in the ideal class group. In particular, show that
   $$P_2P_3P_5 = (\beta)$$
   for some $\beta \in \mathbb{Z}[\sqrt{-21}]$

   Hint: Note that this will require $N(\beta) = 30$ – this should suggest a candidate choice!

3. Childress Exercise 3.5. Based on what we know at this point in the book, this is only meant to be based on **analytic** considerations – is it possible for the Dirichlet densities to align for $\mathcal{H}_1$ and $\mathcal{H}_2$?

4. As mentioned in lecture, the proof of the uniqueness of the class field (Childress Theorem 3.2.2) requires a technical result of primes that split completely. In particular, the claim is that if K_1/F and K_2/F are field extensions, and the compositum is $K := K_1K_2$, then $\mathcal{S}_{K/F} = \mathcal{S}_{K_1/F} \cap \mathcal{S}_{K_2/F}$. In words, if $P \subset \mathcal{O}_F$ is prime, then P splits completely in K if and only if P splits completely in K_1 and K_2.

(a) Embed K in a normal extension L, so that L/F is Galois. If $Q \subset \mathcal{O}_L$ is a prime dividing P, then consider the decomposition group $D = D(Q, P)$, and corresponding fixed field L_D. The Tower Theorem (Ash 8.2.7) implies that P splits completely in L_D. If L' is an intermediate field, let $Q_{L'} := Q \cap \mathcal{O}_{L'}$ be the corresponding prime. Show that $L' \subset L_D$ if and only if $e(Q_{L'}, P) = f(Q_{L'}, P) = 1$.

(b) Apply part (a) to K_1 and K_2. Then conclude that $K \subset L_D$.

Remark: We will later see a local proof of this result.

In Homework 3 Problems 6–9 you showed that $K = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$ is an extension of $F = \mathbb{Q}(\sqrt{-15})$ in which **all** primes are unramified. Problems 5 –7 will show that K is the Hilbert class field for F.

5. In this problem you will work with the class group of F. Note that $\mathcal{O}_{\mathcal{F}} = \mathbb{Z}[\alpha]$, where $\alpha := \frac{1+\sqrt{-15}}{2}$.

   (a) Applying Minkowski's bound, you need only check the factorization of the ideal (2) in order to find the class group. Show that $(2) = (2, \alpha)(2, \alpha + 1) =: P_2 P_2'$.

   (b) Show that the only primes that ramify in $\mathcal{O}_{\mathcal{F}}$ are 3 and 5, and find their factorizations, as $(3) = P_3^2, (5) = P_5^2$.

   (c) By part (a), the class group should be $\mathcal{CL}(F) \cong \{1, P_2\}$. Show that $P_2 P_3$ and $P_2' P_3$ are principal ideals by finding explicit generators.

6. In this problem you will prove that there are no primes $p \in \mathbb{N}$ that are inert in K.

   (a) Consider the three quadratic subfields $F, F_3 := \mathbb{Q}(\sqrt{-3}), F_5 := \mathbb{Q}(\sqrt{5}) \subset K$. Show that p cannot be inert in all three simultaneously.

   (b) Use part (a) to conclude that p cannot be inert in K.

   (c) Determine the set of residue classes

   $$\mathcal{H} := \{a \mid p \text{ splits in } F \iff p \equiv a \pmod{15}\}.$$

   Show that $\mathcal{H}$ is a subgroup of $(\mathbb{Z}/15\mathbb{Z})^\times$.

7. In general, K is the *Hilbert class field* of F if it is the class field over F of $\mathcal{P}_F$, the set of all principal ideals (i.e., the modulus is $\mathcal{M} = (1) = \mathcal{O}_F$).

   (a) Using Problem 6, show that an ideal $P \subset \mathcal{O}_F$ splits completely in K if and only if $P = (p)\mathcal{O}_F$ for a prime p that is inert in F.

   (b) Show that K is the Hilbert class field of F. In fact, in this case the set of splitting primes $\mathcal{S}_{K/F}$ is **equal** to $\mathcal{P}_F$ (the general requirement for a class field is that these may differ by a set of Dirichlet density zero).

   Remark: As mentioned above, K/F is an abelian extension in which all primes are unramified. It is a deep fact that this is **always** *the case for the Hilbert class field, and the definition is often alternatively given as the maximal abelian, unramified extension of F.*