MATH 7230 Homework 6 - Fall 2019

Due Tuesday, Nov. 12 at 10:30

`https://www.math.lsu.edu/~mahlburg/teaching/2019-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

1. Childress Exercise 4.4.6. For example, you will need to show that if a is a limit point of H, then so is a^{-1}.

2. Childress Exercise 4.4.8 (b) and (c). These are short problems that will test your understanding of the definition of topological groups!

3. Childress Exercise 4.4.9. Part (a) is similar to Exercise 4.4.8(a), which was proved in lecture.

   For part (b), it will be helpful to consider number theoretic examples. In $(\mathbb{R}, +)$, the interval $(-1, 1)$ is an open neighborhood of the identity that is **not** a subgroup. However, the statement is true in $(\mathbb{Q}_p, +)$: **any** neighborhood $U(0, \varepsilon) = \{||x||_p < \varepsilon\}$ is in fact a subgroup. Try to characterize these phenomena using topological properties.

4. Suppose that $A \subset \mathbb{Z}_p$ is an open neighborhood, and define the characteristic function

   $$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

   Prove that $\mathbb{1}_A$ is continuous.

   Hint: Show that $\mathbb{1}_A$ is locally constant.

5. In this problem you will prove Hensel's Lemma, which both proves the existence and provides a computational algorithm for finding p-adic roots of polynomials. Suppose that $f(X) = c_k X^k + \cdots + c_1 X + c_0$ is a polynomial in $\mathbb{Z}_p[X]$. Furthermore, suppose that $\alpha \in \mathbb{Z}_p$ satisfies $||f(\alpha)||_p < 1$ and $||f'(\alpha)||_p = 1$. Set $\alpha_0 := \alpha$, and for $n \geq 1$,

   $$\alpha_{n+1} := \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}.$$

   The claim is then that $f(\alpha_n) \to 0$. More precisely, the rate of convergence is at least $||f(\alpha_n)||_p < \frac{1}{p^n}$.

   (a) Use the Binomial theorem to show that so long as $f(X)$ is in a ring R where $(j!)^{-1}$ exists for $j = 1, 2, \ldots, m$, then a version of "Taylor series" holds:

   $$f(x + y) = f(x) + x \cdot f'(y) + x^2 \frac{f''(y)}{2!} + \cdots + x^j \frac{f^{(j)}(y)}{j!} + O\left(x^{j+1}\right).$$

   Here the big-O notation is simply shorthand for a term in $x^{j+1} R[x, y]$.

(b) Apply part (a) to the p-adic case, concluding that

$$f(\alpha_{n+1}) = f(\alpha_n) - \frac{f(\alpha_n)}{f'(\alpha_n)} \cdot f'(\alpha_n) + O\left(\frac{f(\alpha_n)^2}{f'(\alpha_n)^2}\right).$$

(c) Use the ultrametric inequality and induction to conclude the proof.

Remark: Note that Hensel's Lemma is **very** *similar to Newton's Method for finding roots of real polynomials.*

6. (a) Childress Exercise 4.14. The infinite places will require the most thought.

(b) Childress Exercise 4.15. Here the infinite places do not play a role. You need to show that for each $\alpha \in F^\times$ there is an open set $\iota(\alpha) \in U_\alpha$ such that $\iota(\alpha') \notin U_\alpha$ for any $\alpha' \neq \alpha$.

In Problems 7 – 9 you will complete Childress Exercise 4.16, which outlines the *idelic* proof that the class group (of a number field) is finite.

7. The most involved step (by far) is to prove that $J_F^1/F^\times$ is compact, which will occupy this problem and the next. As these are very lengthy calculations, you are encouraged to work through at least some of them if you are not able to finish them all. Recall that

$$J_F^1 := \ker(\text{content}) = \Big\{(a_\nu) \mid \prod_{\nu \in V_F} ||a_\nu||_\nu = 1\Big\}.$$

For your reference, the argument outlined here largely follows T. Weston's notes:

`http://people.math.umass.edu/~weston/oldpapers/idele.pdf`.

(a) If $\boldsymbol{a} = (a_\nu)_\nu$ is an idéle, define

$$L(\boldsymbol{a}) := \left\{\alpha \in F^\times \mid ||\alpha||_\nu \leq ||a_\nu||_\nu \ \forall v \in V_F\right\}.$$

Furthermore, let $\ell(\boldsymbol{a}) := \#L(\boldsymbol{a})$. Use a simple bijection in order to prove that if $\ell(\boldsymbol{a}) = \ell(\beta\boldsymbol{a})$.

Remark: Throughout these problems $||\alpha||_\nu$ is written instead of Childress' more careful notation $||\iota_\nu(\alpha)||_\nu$ (which is used as a reminder that the local embeddings might involve non-canonical choices).

(b) The first goal is to show that there is a constant $c_F \in \mathbb{R}_{>0}$ such that for any $\boldsymbol{a} \in J_F$,

$$\ell(\boldsymbol{a}) \geq c_F \cdot \text{content}(\boldsymbol{a}). \tag{1}$$

Suppose that the ring of integers has a basis $\mathcal{O}_F = \langle b_1, \ldots, b_n \rangle_{\mathbb{Z}}$, and set

$$c_1 := n \cdot \sup_{\substack{1 \leq i \leq n \\ \nu | \infty}} \{||b_i||_\nu\}.$$

Prove that there is some $\alpha \in F^\times$ such that

$$\frac{c_1}{||a_\nu||_\nu} \leq ||\alpha||_\nu \leq \frac{2c_1}{||a_\nu||_\nu} \quad \text{for all } \nu \mid \infty.$$

Furthermore, show that there is some $m \in \mathbb{N}$ such that $||m\alpha a_\nu||_\nu \leq 1$ for all $\nu < \infty$.

(c) Combine parts (a) and (b) to conclude that without loss of generality (in terms of trying to prove (1)), you can assume that $||a_\nu||_\nu \le 1$ for all $\nu < \infty$, and

$$mc_1 \le ||a_\nu||_\nu \le 2mc_1 \quad \text{for all } \nu \mid \infty. \tag{2}$$

(d) Use part (c) to show that $I_{\boldsymbol{a}} := \eta(\boldsymbol{a}) = \prod_{\nu<\infty} P_\nu^{\mathrm{ord}_\nu(a_\nu)}$ is an ideal in $\mathcal{O}_F$. Let

$$B_m := \{k_1 b_1 + \cdots + k_n b_n \mid 0 \le k_j \le m\},$$

and note that $\#B_m = (m+1)^n$. Define the quotient map $\varphi : B_m \to \mathcal{O}_F/I_{\boldsymbol{a}}$. Use a pigeonhole argument to to show that there is some fiber

$$\left|\varphi^{-1}(\boldsymbol{y} + I_{\boldsymbol{a}})\right| \ge \frac{(m+1)^n}{N(I_{\boldsymbol{a}})}.$$

Show that $\boldsymbol{x} \in \varphi^{-1}(\boldsymbol{y} + I_{\boldsymbol{a}})$, then $\boldsymbol{x} - \boldsymbol{y} \in L(\boldsymbol{a})$, and conclude that

$$\ell(\boldsymbol{a}) \ge \frac{(m+1)^n}{N(I_{\boldsymbol{a}})}. \tag{3}$$

(e) Finally, use (2) and (3) to conclude that (1) holds with $c_F := (2c_1)^{-n}$.
Remark: This problem may be considered an idelic version of Minkowski's Theorem (and indeed, the set $L(\boldsymbol{a})$ is essentially a subset of a lattice over the infinite places, and the Geometry of Numbers shows that $c_F = 2^{r_1}(2\pi)^{r_2}/\sqrt{\mathrm{Disc}(F)}$ suffices). However, the idelic approach gives a very different perspective.

8. This is a continuation of Problem 7. Throughout this problem the content will be written as $c(\boldsymbol{a})$.

(a) Suppose that $c(ba) \ge \frac{2}{c_F}$. Prove that there is some $\alpha \in F^\times$ such that

$$1 \le ||\alpha a_\nu||_\nu \le c(\boldsymbol{a}) \quad \text{for all } \nu. \tag{4}$$

(b) Use the fact that J_F is a topological group to show that $c^{-1}(\rho)$ is homeomorphic to J_F^1 for any $\rho \in \mathbb{R}_{>0}$.

(c) Pick $\rho > \frac{2}{c_F}$ and $\boldsymbol{a} \in c^{-1}(\rho)$. Use part (a) to show that there is $\alpha \in F^\times$ such that

$$1 \le ||\alpha a_\nu||_\nu \le \rho \qquad \text{for all } \nu.$$

Then argue that there must be a finite set $S \subset V_F$ such that

$$\begin{cases} 1 \le ||\alpha a_\nu||_\nu \le \rho & \text{if } \nu \in S, \\ ||\alpha a_\nu||_\nu = 1 & \text{if } \nu \notin S. \end{cases}$$

(d) Finally, define

$$T := \prod_{\nu \in S} \left(\overline{U(0,\rho)} \setminus U(0,1)\right) \prod_{\nu \notin S} \mathcal{U}_\nu.$$

Since each factor is compact, Tychonoff's Theorem implies that T is compact. Show that $c^{-1}(\rho)/F^\times \subset T/F^\times$. Since $c^{-1}(\rho)/F^\times$ is closed, conclude that it is compact. Use part (b) to conclude that $J_F^1/F^\times$ is compact.

9. Prove Childress Exercise 4.16 (b), (c), and (d). Part (b) should remind you of Problem 6.