MATH 7230 Homework 7 - Fall 2019

Due Tuesday, Nov. 26 at 10:30

`https://www.math.lsu.edu/~mahlburg/teaching/2019-MATH7230.html`

You are required to turn in at least **one** of the following problems, and must complete a total of **20** by semester's end. Group work is allowed, but your solutions must be written up individually.

1. (a) Childress Exercise 5.1.
   (b) Explain why this implies that the conductor is well-defined; i.e., that there exists a unique $\mathfrak{f}(K/F) \subset \mathcal{O}_F$ such that $\mathcal{E}^+_{F,\mathfrak{f}} \subset \mathcal{H}$.

2. Childress Exercise 5.3. You should use the fact that $N_{K/F} = N_{E/F} \circ N_{K/E}$ to show that $\mathcal{H}_K \subset \mathcal{H}_E$. The relation between $\mathfrak{f}(K/F)$ and $\mathfrak{f}(E/F)$ then follows quickly.

In Problems 3 – 4 you will explore Proposition 5.1.1 (Childress' "Consistency" result) in a familiar, in-depth example (recall HW 3 Problems 6–9 and HW 5 Problems 5–7).

Let $K := \mathbb{Q}(\sqrt{-3}, \sqrt{5}), E := \mathbb{Q}(\sqrt{-3}), L := \mathbb{Q}(\sqrt{5}), F = \mathbb{Q}$. Define the automorphisms $\sigma_3 : \sqrt{-3} \mapsto -\sqrt{-3},\ \sigma_5 : \sqrt{5} \mapsto -\sqrt{5}$; then $G = \mathrm{Gal}(K/F) = \{1, \sigma_3, \sigma_5, \sigma_3\sigma_5\}$. In a slight abuse of notation (i.e. identifying $\sigma_3 \mid_L$ with σ_3), then $\mathrm{Gal}(E/F) = \langle \sigma_3 \rangle$, $\mathrm{Gal}(L/F) = \langle \sigma_5 \rangle$.

3. Consider the prime ideal $P = (11) \subset \mathbb{Z} = \mathcal{O}_F$.
   (a) Find the factorization of P throughout the field tower.
   (b) Correctly identify the Artin symbol $\left(\dfrac{P}{K/F}\right)$ as an element of G.
   (c) Next, find $\left.\left(\dfrac{P}{K/F}\right)\right|_L$ as an element of $\mathrm{Gal}(L/F)$, and compare to $\left(\dfrac{P}{L/F}\right)$. Verify that Corollary 5.1.2 holds.
   (d) Finally, identify $\left(\dfrac{P_E}{K/E}\right)$. Then calculate $N_{E/K}(P_E)$, and verify that Corollary 5.1.3 holds.

4. (a) Characterize the primes $P = (p) \subset \mathbb{Z}$ such that $\left(\dfrac{P}{K/F}\right) = \tau$ for each $\tau \in G$. Your answer should be in terms of congruence conditions.
   (b) If you are feeling ambitious, carry out Problem 3 for a different prime.

   Remark: You might also find it instructive to instead let $K = \mathbb{Q}(\zeta_{15})$ (which is a quadratic extension of the above K), and then perform all calculations in $\mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/15\mathbb{Z})^\times$, so that all Galois automorphisms are of the form $\sigma_a : \zeta_{15} \mapsto \zeta_{15}^a$.

In Problems 5 – 6 you will prove that the absolute norm map is surjective for local field extensions (this was needed in the proof that the only divisors of the conductor are ramified primes). For simplicity, you will only consider extensions of $\mathbb{Z}_p$.

5. First, consider the finite field extension $\mathbb{F}_{p^n}/\mathbb{F}_p$. Recall that the Galois group is $G = \langle\sigma_p\rangle$, where $\sigma_p : \alpha \mapsto \alpha^p$ is the Frobenius element. In particular, $G = \{1, \sigma_p, \ldots, \sigma_p^{n-1}\}$.

   (a) Show that the norm map $N_{\mathbb{F}_{p^n}/\mathbb{F}_p} : \mathbb{F}_{p^n} \to \mathbb{F}_p$ is surjective. Recall that for a Galois extension K/F, the (absolute) norm is $N_{K/F} := \displaystyle\prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma$.

   (b) Now show that the trace map $\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}$ is also surjective. Recall that the trace is given by $\mathrm{Tr}_{K/F} := \displaystyle\sum_{\sigma \in \mathrm{Gal}(K/F)} \sigma$.

   Hint: Use the linear independence of field automorphisms (as in Ash Proposition 2.1.8). This shows that the image of the trace is strictly larger than $\{0\}$ *– why can you then conclude that it is* **all** *of* $\mathbb{F}_p$*?*

6. Suppose that $K/\mathbb{Q}$ is a Galois extension, and consider a place $\omega \mid p$, with corresponding local field extension $K_\omega/\mathbb{Q}_p$, and residue field $K_\omega/(p)\mathcal{O}_\sqsupseteq \cong \mathbb{F}_{p^n}$. The claim is then that $N_{K_\omega/\mathbb{Q}_p} : \mathcal{O}_\omega^\times \to \mathbb{Z}_p^\times$ is surjective.

   (a) Use Problem 5 (a) to show that it is sufficient to show that the image of the norm map contains $1 + p\mathbb{Z}_p$.

   (b) Now suppose that $a_1 \in \mathbb{F}_p$. Use Problem 5 (b) to show that there is some $b_1 \in F_{p^n}$ such that
   $$N_{K_\omega/\mathbb{Q}_p}(1 + b_1 p) \equiv 1 + a_1 p \pmod{p^2\mathbb{Z}_p}.$$

   (c) Continuing recursively from part (b), prove that for any $\alpha = 1 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p$, there is $\beta \in \mathcal{O}_\omega^\times$ such that $N_{K_\omega/\mathbb{Q}_p}(\beta) = \alpha$.