IMPROPER INTEGRALS

This document briefly discusses improper integrals of the second kind, which are integrals on a finite range whose values go to ∞ (improper integrals of the first kind are those for which the domain goes to ∞). For example,

$$\int_0^1 \frac{1}{x}\,dx$$

is improper, because $\lim_{x\to 0^+} \frac{1}{x} = \infty$. Instead, such integrals must be defined as limits; in the previous case

$$\int_0^1 \frac{1}{x}\,dx := \lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx.$$

In general, improper integrals of the second kind require more manipulation and close approximation than the first kind, because the notion of asymptotics does not come into play. The following result is the main tool used to understand improper integrals of the second kind.

Fact 1. *If p is a real number, then the integral*

$$\int_0^1 \frac{1}{x^p}\,dx \qquad \begin{cases} \text{Converges} & \text{if } p < 1, \\ \text{Diverges} & \text{if } p \geq 1. \end{cases}$$

Example 1. Determine whether the following integral converges:

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx.$$

Solution. This is an improper integral of the second kind because the denominator is 0 at $x = 1$. One method for solving this problem is to notice that the polynomial factors, so the integration term is

$$\frac{1}{\sqrt{1-x^4}} = \frac{1}{\sqrt{(1-x)(1+x+x^2+x^3)}}.$$

For the integral, $0 \leq x \leq 1$, so $1 \leq 1+x+x^2+x^3 \leq 4$. This allows a simple comparison:

$$\int_0^1 \frac{1}{\sqrt{(1-x)(1+x+x^2+x^3)}}\,dx \leq \int_0^1 \frac{1}{\sqrt{(1-x)\cdot 1}}\,dx. \tag{1}$$

Finally, substitute $u = 1 - x$ to rewrite the integral as

$$\int_1^0 \frac{-1}{\sqrt{u}}\,du = \int_0^1 \frac{1}{\sqrt{u}}\,du.$$

This last integral has exponent $1/2$, so it converges.

An alternative method is to make a more direct comparison, which is built sequentially in order to keep track of signs and inequalities.

$$\begin{aligned}
x^4 &\leq x && \text{(For } 0 \leq x \leq 1\text{)} \\
1 - x^4 &\geq 1 - x && \\
\sqrt{1 - x^4} &\geq \sqrt{1 - x} && \text{(Square root preserves inequalities)} \\
\frac{1}{\sqrt{1 - x^4}} &\leq \frac{1}{\sqrt{1 - x}} && \text{(Inversion switches inequalities).}
\end{aligned}$$

This last line again shows that the original integral is less than the integral of $\frac{1}{\sqrt{1-x}}$, just like in equation (1). $\square$

Both methods have their limitations – the first requires a polynomial that can be easily factored and the second method may require a change of variables, as demonstrated in the next example.

Example 2. Use a comparison to show that the following integral converges:

$$\int_0^3 \frac{1}{\sqrt[3]{36 - 3x - 3x^2}}\, dx.$$

Solution. This is an improper integral because $36 - 3x - 3x^2$ is zero when $x = 3$. Since $x \leq 3$, a first approximation is thus

$$\int_0^3 \frac{1}{\sqrt[3]{36 - 3x - 3x^2}}\, dx \leq \int_0^3 \frac{1}{\sqrt[3]{27 - 3x^2}}\, dx.$$

To use the second method from above, we would like to replace x^2 by x, but can no longer do this directly because here $0 \leq x \leq 3$, where it is not true that $x^2 \leq x$. However, if we substitute $x = 3u$, then we have

$$\int_0^3 \frac{1}{\sqrt[3]{27 - 3x^2}}\, dx = \int_0^1 \frac{3}{\sqrt[3]{27 - 27u^2}}\, du = \int_0^1 \frac{1}{\sqrt[3]{1 - u^2}}\, du.$$

Now we may proceed as before, making the comparison

$$\int_0^1 \frac{1}{\sqrt[3]{1 - u^2}}\, du \leq \int_0^1 \frac{1}{\sqrt[3]{1 - u}}\, du.$$

$\square$

There is an important class of improper integrals for which determining the behavior is relatively simple.

Fact 2. *If an improper integral of the second kind is factored as*

$$\int_a^b \frac{f(x)}{(x - c)^k g(x)}\, dx,$$

where $a \le c \le b$ and $f(x)$ and $g(x)$ do not have any zeroes in the range $[a, b]$, then the convergence is dictated by the exponent k (in precisely the same way as the exponent p in Fact 1).

To show the power of this result, recall that in Example 1, we had

$$\frac{1}{\sqrt{1-x^4}} = \frac{1}{\sqrt{(1-x)(1+x+x^2+x^3)}} = \frac{1}{(1-x)^{\frac{1}{2}}(1+x+x^2+x^3)^{\frac{1}{2}}}.$$

Therefore, the relevant exponent is $k = \frac{1}{2}$, and it converges! This also works for Example 2, since

$$\frac{1}{\sqrt[3]{36-3x-3x^2}} = \frac{1}{\sqrt[3]{(12+3x)(3-x)}} = \frac{1}{(3-x)^{\frac{1}{3}}(12+3x)^{\frac{1}{3}}}.$$

Now the exponent is $k = \frac{1}{3}$.

Example 3. Determine whether the following integral converges:

$$\int_1^2 \frac{x^3+8}{(1-x^2)\sqrt{3-2x-x^2}}.$$

Solution. Factor all terms to find that

$$\begin{aligned}\frac{x^3+8}{(1-x^2)\sqrt{3-2x-x^2}} &= \frac{(x+2)(x^2-2x+4)}{(1-x)(1+x)\sqrt{(1-x)(3+x)}} \\ &= \frac{(x+2)(x^2-2x+4)}{(1-x)^{\frac{3}{2}}(1+x)(3+x)^{\frac{1}{2}}}.\end{aligned}$$

The only zero that lies in the range $[1, 2]$ is from the term $\frac{1}{(1-x)^{3/2}}$, so Fact 2 applies. Since the exponent is $\frac{3}{2} \ge 1$, the integral diverges. $\square$

Finally, as food for thought, determine what can be said about this integral (note that the conditions of Fact 2 do not apply):

$$\int_0^3 \frac{x(x-2)}{\sqrt{(x-1)(x-3)}}.$$

Hint: Try breaking up the interval.