LINEAR ASYMPTOTICS EXAMPLE

This document briefly reviews linear asymptotics for rational functions. Suppose that $f(x) = p(x)/q(x)$ is a rational function, where both $p(x)$ and $q(x)$ are polynomials. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$, then $f(x)$ will have a *linear asymptote* that can be found by dividing p by q. This is illustrated by a simple example.

Example. Let
$$f(x) = \frac{3x^2}{2x+5},$$
which is a rational function whose numerator is degree 2, and whose denominator is degree 1. Polynomial division shows that
$$3x^2 = \left(\frac{3x}{2} - \frac{15}{4}\right) \cdot (2x+5) + \frac{75}{4},$$
so
$$f(x) = \left(\frac{3x}{2} - \frac{15}{4}\right) + \frac{75}{4} \cdot \frac{1}{2x+5}.$$

The first two terms of this expansion describe a line, and this is the linear asymptote of $f(x)$, which just means that as x grows, the "remainder" term becomes unimportant. The more formal way to state this is that
$$\lim_{x\to\infty} \frac{f(x)}{3x/2 - 15/4} = 1.$$

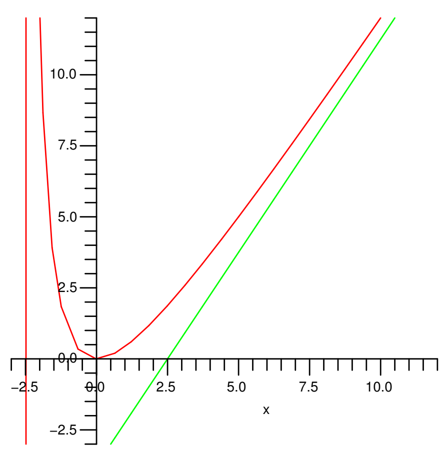

FIGURE 1. Graph of $f(x) = \frac{3x^2}{2x+5}$ and $y = \frac{3x}{2} - \frac{15}{4}$

Remark. In the previous example, the difference between $f(x)$ and its asymptote was small, since

$$f(x) - \left(\frac{3x}{2} - \frac{15}{4}\right) = \frac{75}{4} \cdot \frac{1}{2x+5},$$

which goes to zero as x goes to ∞. However, in general it's only necessary that the difference between $f(x)$ and its asymptote by *relatively* small. This is illustrated by

$$g(x) = 2x + \sqrt{3 + |x|\ } \cdot \sin x.$$

This function behaves erratically for small values of x, but

$$\lim_{x\to\infty} \frac{g(x)}{2x} = 1.$$

This is in sharp contrast to the previous example, for the difference between $g(x)$ and it's asymptote can still be large:

$$g(x) - 2x = \sqrt{3 + |x|\ } \cdot \sin x.$$

The asymptote is still valid though, because this error is approximately $\sqrt{x}$, which is dwarfed by $2x$ for large x.