Adaptive Tracking and Parameter Identification for Nonlinear Control Systems

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Holder of Roy Paul Daniels Professorship #3
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Department of Mathematics Colloquium
LSU – September 13, 2012
What Do We Mean By Control Systems?

These are doubly parameterized families of ODEs of the form

\[
\dot{Y} = F(t, Y, u(t, Y), \delta(t)), \quad Y \in \mathbb{R}^n.
\]

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We have freedom to choose the control function \( u(t, Y) \).

The functions \( \delta: [0, \infty) \to D \) represent uncertainty.

\( D \subseteq \mathbb{R}^m \).

Specify \( u(t, Y) \) to get a singly parameterized family

\[
\dot{Y} = G(t, Y, \delta(t)), \quad Y \in \mathbb{R}^n,
\]

(2)

where \( G(t, Y, d) = F(t, Y, u(t, Y), d) \).

Typically we construct \( u(t, Y) \) so that all trajectories of (2) for all possible choices of \( \delta \) satisfy some control objective.
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Find \(\gamma_i\)'s by building certain strict LF\(s\) for \(\dot{Y} = G(t, Y, 0)\).
What Makes a LF Nonstrict or Strict?

A LF for $\dot{Y} = G(t,Y)$ is a proper positive definite $C^1$ function $V$ that admits a positive semidefinite function $W$ such that $V_t(t,Y) + V_Y(t,Y) G(t,Y) \leq -W(Y)$ for all $t \geq 0$ and $Y \in Y$.

If, in addition, $W$ is positive definite, then we call $V$ strict.

Proper positive definite on $Y = \mathbb{R}^n$: $\exists \alpha_i \in \mathbb{K} \infty$ such that $\alpha_1(|Y|) \leq V(t,Y) \leq \alpha_2(|Y|)$ for all $t \geq 0$ and $Y \in Y$.

Positive definiteness (resp., semidefiniteness): 0 at zero and positive (resp., nonnegative) at all other points in $Y$.

Example 1: $\dot{y}_1 = y_2$, $\dot{y}_2 = -y_1 - y_3^2$. $V(Y) = 0$. $\dot{V} = -y_4^2$.

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What is Strictification?

This is the transformation of a nonstrict LF \( V \) into a strict LF \( V^{\#} \) on its domain, and is the subject of my book. Doing so can often strengthen a UGAS result into an ISS result to quantify the effects of uncertainties and robustify controllers. The required nondegeneracy of \( V \) is often expressed in terms of Jurdjevic-Quinn, LaSalle, or Matrosov conditions. Active magnetic bearings, adaptive systems, bioreactors, brushless DC motors, heart rate controllers, marine robots, microelectromechanical relays, systems with control delays, underactuated ships, unmanned air vehicles,..
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Adaptive Tracking and Parameter Identification

Consider a suitably regular nonlinear system

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Problem: Find a dynamic feedback with estimator

\[
u(t, \xi, \hat{\Gamma}), \quad \hat{\Gamma} = \tau(t, \xi, \hat{\Gamma})
\]  

(4)

that makes the \( Y = (\hat{\Gamma}, \tilde{\xi}) = (\Gamma - \hat{\Gamma}, \xi - \xi_R) \) system UGAS.
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$$u(t, \xi, \hat{\Gamma}), \quad \dot{\hat{\Gamma}} = \tau(t, \xi, \hat{\Gamma})$$  \hspace{1cm} (4)

that makes the $Y = (\hat{\Gamma}, \tilde{\xi}) = (\Gamma - \hat{\Gamma}, \xi - \xi_R)$ system UGAS.

Flight control, electrical and mechanical engineering, etc.
Adaptive Tracking and Parameter Identification

Consider a suitably regular nonlinear system

\[ \dot{\xi} = \mathcal{J}(t, \xi, \Gamma, u) \]  

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**Persistent excitation.**
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Persistent excitation. Annaswamy, Narendra, Teel..
New Results (Mazenc, de Queiroz, M., ’11)
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We solved the tracking and parameter identification problem for

\[
\begin{align*}
\dot{x} &= f(\xi) \\
\dot{z}_i &= g_i(\xi) + k_i(\xi)\theta_i + \psi_i u_i, \quad i = 1, 2, \ldots, s.
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where \(\lambda_i(t) = (k_i(\xi_R(t)), \dot{z}_{R,i}(t) - g_i(\xi_R(t)))\) for \(i = 1, 2, \ldots, s.\)
Two Other Key Assumptions
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- We know $v_f$ and a strict LF $V : [0, \infty) \times \mathbb{R}^{r+s} \to [0, \infty)$ for

$$\begin{cases} 
\dot{X} &= f((X, Z) + \xi_R(t)) - f(\xi_R(t)) \\
\dot{Z} &= v_f(t, X, Z)
\end{cases}$$

(7)

such that $-\dot{V}$ and $V$ have positive definite quadratic lower bounds near 0,
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  \[
  \underline{\psi} < \psi_i < \overline{\psi} \quad \text{and} \quad |\theta_i| < \theta_M
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  \]
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where \(\lambda_i(t) = (k_i(\xi_R(t)), \dot{z}_R,i(t) - g_i(\xi_R(t)))\) for \(i = 1, 2, \ldots, s.\)
Dynamic Feedback

The estimator evolves on \( \prod_{i=1}^{s} (\theta^M, \theta^M) \times (\psi, \psi) \times s \).

\[
\dot{\hat{\theta}}_{ij} = (\hat{\theta}_{ij}^2 - \theta^M_{ij}) \varpi_{ij}, \quad 1 \leq i \leq s, \quad 1 \leq j \leq p_i
\]

\[
\dot{\hat{\psi}}_i = (\hat{\psi}_i - \psi)(\hat{\psi}_i - \psi) \Upsilon_i, \quad 1 \leq i \leq s
\]

Here \( \hat{\theta}_i = (\hat{\theta}_i^1, ..., \hat{\theta}_i^{p_i}) \) for \( i = 1, 2, ..., s \), \( \varpi_{ij} = -\frac{\partial V}{\partial \tilde{z}_i(t, \tilde{\xi})} \) \( k_{ij}(\tilde{\xi} + \xi R(t)) \) and \( \Upsilon_i = -\frac{\partial V}{\partial \tilde{z}_i(t, \tilde{\xi})} u_i(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}) \).

\[
u_i(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}) = v_{f, i}(t, \tilde{\xi}) - g_i(\xi) - k_i(\xi) \hat{\theta}_i + \dot{z}_R, s_i(t) \hat{\psi}_i
\]
Dynamic Feedback

The estimator evolves on \( \{ \prod_{i=1}^{s} (-\theta_M, \theta_M)^{p_i} \} \times \overline{\psi}^s \).
Dynamic Feedback

The estimator evolves on \(\{\prod_{i=1}^{s}(-\theta_M, \theta_M)^{p_i}\} \times (\psi, \bar{\psi})^s\).

\[
\begin{align*}
\dot{\hat{\theta}}_{i,j} &= (\hat{\theta}_{i,j}^2 - \theta_M^2) \varpi_{i,j}, \quad 1 \leq i \leq s, \ 1 \leq j \leq p_i \\
\dot{\hat{\psi}}_i &= (\hat{\psi}_i - \psi) (\hat{\psi}_i - \bar{\psi}) \gamma_i, \quad 1 \leq i \leq s 
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\]
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The estimator evolves on \( \prod_{i=1}^{s} (-\theta_{M}, \theta_{M})^{p_{i}} \times (\psi, \bar{\psi})^{s} \).

\[
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\dot{\hat{\theta}}_{i,j} &= (\hat{\theta}_{i,j} - \theta_{M}^{2}) \varpi_{i,j}, \quad 1 \leq i \leq s, 1 \leq j \leq p_{i} \\
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\[
\varpi_{i,j} = -\frac{\partial V}{\partial \tilde{z}_i}(t, \tilde{\xi}) k_{i,j}(\tilde{\xi} + \xi_R(t))
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The estimator evolves on \( \prod_{i=1}^{s} (-\theta_M, \theta_M)^{p_i} \times (\psi, \overline{\psi})^s \).

\[
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\]

(9)

Here \( \hat{\theta}_i = (\hat{\theta}_{i,1}, \ldots, \hat{\theta}_{i,p_i}) \) for \( i = 1, 2, \ldots, s \),

\[
\varpi_{i,j} = -\frac{\partial V}{\partial \tilde{z}_i}(t, \tilde{\xi})k_{i,j}(\tilde{\xi} + \xi_R(t)) \quad \text{and} \\
\gamma_i = -\frac{\partial V}{\partial \tilde{z}_i}(t, \tilde{\xi})u_i(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}).
\]

(10)
Dynamic Feedback

The estimator evolves on \( \prod_{i=1}^{s} (-\theta_M, \theta_M)^{p_i} \times (\underline{\psi}, \overline{\psi})^s \).

\[
\begin{cases}
\dot{\hat{\theta}}_{i,j} &= (\hat{\theta}_{i,j}^2 - \theta_M^2) \omega_{i,j}, \quad 1 \leq i \leq s, 1 \leq j \leq p_i \\
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\[
u_{f,i}(t, \tilde{\xi}) - g_i(\xi) - k_i(\xi) \hat{\theta}_i + \dot{z}_{R,i}(t) \overline{\psi}_i
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The estimator evolves on \( \prod_{i=1}^{S} (-\theta_M, \theta_M)^{p_i} \times (\psi, \psi)^{s} \).

\[
\begin{cases}
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\]

\[
u_f, i(t, \tilde{\xi}) = \frac{g_i(\xi) - k_i(\xi) \dot{\theta} + \dot{z}_{R,i}(t)}{\dot{\psi}_i} 
\]

The estimator and feedback can only depend on things we know.
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Augmented Error Dynamics

\[
\begin{align*}
\dot{\tilde{x}} &= f(\tilde{\xi} + \xi_R(t)) - f(\xi_R(t)) \\
\dot{\tilde{z}}_i &= v_{f,i}(t, \tilde{\xi}) + k_i(\tilde{\xi} + \xi_R(t))\tilde{\theta}_i \\
&\quad + \tilde{\psi}_i u_i(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}), \quad 1 \leq i \leq s \\
\dot{\tilde{\theta}}_{i,j} &= -\left(\hat{\theta}_{i,j}^2 - \theta_M^2\right)\omega_{i,j}, \quad 1 \leq i \leq s, 1 \leq j \leq p_i \\
\dot{\tilde{\psi}}_i &= -\left(\hat{\psi}_i - \psi\right)\left(\hat{\psi}_i - \bar{\psi}\right)\gamma_i, \quad 1 \leq i \leq s.
\end{align*}
\tag{12}
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(12)

Tracking error: \(\tilde{\xi} = (\tilde{x}, \tilde{z}) = \xi - \xi_R = (x - x_R, z - z_R)\)

Parameter estimation errors: \(\tilde{\theta}_i = \theta_i - \hat{\theta}_i\) and \(\tilde{\psi}_i = \psi_i - \hat{\psi}_i\)
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\[\mathcal{Y} = \mathbb{R}^{r+s} \times \left(\prod_{i=1}^{s} \left\{\prod_{j=1}^{p_i} (\theta_{i,j} - \theta_M, \theta_{i,j} + \theta_M)\right\}\right) \times \left(\prod_{i=1}^{s} (\psi_i - \overline{\psi}, \psi_i - \overline{\psi})\right).\]
Two Other Key Assumptions

- We know $v_f$ and a strict LF $V : [0, \infty) \times \mathbb{R}^{r+s} \rightarrow [0, \infty)$ for

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for each $i \in \{1, 2, \ldots, s\}$. Known directions for the $\psi_i$'s.
Stabilization Analysis

We build a strict LF for the augmented tracking and identification vector \( Y = (\tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = (\xi - \xi R, \theta - \hat{\theta}, \psi - \hat{\psi}) \) dynamics on \( Y \).

We start with this nonstrict barrier type LF on \( Y \):

\[
V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = V(t, \tilde{\xi}) + s \sum_{i=1}^{p} \sum_{j=1}^{M} \int_{0}^{\theta_i,j} m_{\theta}^2 - (m_{\theta} - \theta_i,j)^2 \, dm + s \sum_{i=1}^{p} \int_{0}^{\psi_i} m(\psi - \psi_i + m_{\psi}) (\psi - \psi_i) \, dm.
\]

On \( Y \),

\[
\dot{V}_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) \leq -W(\tilde{\xi})
\]

for some positive definite function \( W \).

We transform \( V_1 \) into the desired strict LF.
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We build a strict LF for the augmented tracking and identification vector $Y = (\tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = (\xi - \xi_R, \theta - \hat{\theta}, \psi - \hat{\psi})$ dynamics on $\mathcal{Y}$. 
Stabilization Analysis

We build a strict LF for the augmented tracking and identification vector \( Y = (\tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = (\xi - \xi_R, \theta - \hat{\theta}, \psi - \hat{\psi}) \) dynamics on \( \mathcal{Y} \).

We start with this nonstrict barrier type LF on \( \mathcal{Y} \):

\[
V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = V(t, \tilde{\xi}) + \sum_{i=1}^{s} \sum_{j=1}^{p_i} \int_{0}^{\tilde{\theta}_{i,j}} \frac{m}{\theta^2_M - (m - \theta_{i,j})^2} \, dm + \sum_{i=1}^{s} \int_{0}^{\tilde{\psi}_i} \frac{m}{(\psi_i - m - \underline{\psi})(\overline{\psi} - \psi_i + m)} \, dm.
\]
Stabilization Analysis

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\]

On \( \mathcal{Y} \), \( \dot{V}_1 \leq -W(\tilde{\xi}) \) for some positive definite function \( W \).
Stabilization Analysis

We build a strict LF for the augmented tracking and identification vector \( Y = (\tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = (\xi - \xi_R, \theta - \hat{\theta}, \psi - \hat{\psi}) \) dynamics on \( \mathcal{Y} \).

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\]

On \( \mathcal{Y} \), \( \dot{V}_1 \leq -W(\tilde{\xi}) \) for some positive definite function \( W \).

We transform \( V_1 \) into the desired strict LF.
Our Transformation (M. et al, ’11)
Our Transformation (M. et al, ’11)

**Theorem:** We can construct \( \mathcal{L} \in \mathcal{K}_\infty \cap \mathcal{C}^1 \) such that

\[
V^\#(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) \overset{\text{def}}{=} \mathcal{L}(V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi})) + \sum_{i=1}^{s} \Omega_i(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) ,
\]

where

\[
\Omega_i(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = -\tilde{z}_i \lambda_i(t) \alpha_i(\tilde{\theta}_i, \tilde{\psi}_i)
\]

\[
+ \frac{1}{T_\psi} \alpha_i^\top(\tilde{\theta}_i, \tilde{\psi}_i) \Omega_i(t) \alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) ,
\]

\[
\alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) = \begin{bmatrix}
\tilde{\theta}_i \psi_i - \theta_i \tilde{\psi}_i \\
\tilde{\psi}_i
\end{bmatrix} , \quad \text{and}
\]

\[
\Omega_i(t) = \int_{t-T}^{t} \int_m \lambda_i^\top(s) \lambda_i(s) ds \, dm ,
\]

is a strict LF for the \( Y = (\tilde{\xi}, \tilde{\theta}, \tilde{\psi}) \) dynamics on \( \mathcal{Y} \), so it is UGAS.
Application: Marine Robots (with Georgia Tech)
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\[ \rho = |r_2 - r_1|, \quad \phi = \text{angle between } x_1 \text{ and } x_2, \quad \cos(\phi) = x_1 \cdot x_2 \]
Application: Marine Robots (with Georgia Tech)

\[ \rho = |\mathbf{r}_2 - \mathbf{r}_1|, \quad \phi = \text{angle between } \mathbf{x}_1 \text{ and } \mathbf{x}_2, \quad \cos(\phi) = \mathbf{x}_1 \cdot \mathbf{x}_2 \]
Curve Tracking Dynamics

\begin{align}
\dot{\rho} &= -\sin(\phi) \\
\dot{\phi} &= \kappa \cos(\phi) + \kappa \rho - u_b,
\end{align}
\begin{align}
\rho, \phi &\in (0, +\infty) \times (-\pi/2, \pi/2) \\
\end{align}
\begin{align}
u_b &= \kappa \cos(\phi) + \kappa \rho - h'(\rho) \cos(\phi) + \mu \sin(\phi)
\end{align}
\begin{align}h(\rho) &= \alpha \{\rho + \rho^2 - 2\rho^3\}, \rho_0 = \text{desired value for} \rho
\end{align}
\begin{align}V(\rho, \phi) &= -\ln(\cos(\phi)) + h(\rho)
\end{align}
\begin{align}U(\rho, \phi) &= -h'(\rho) \sin(\phi) + \frac{1}{\mu} \int_0^\Gamma_0 V(\rho, \phi) d\mu
\end{align}
Curve Tracking Dynamics

\[
\begin{align*}
\dot{\rho} &= -\sin(\phi) \\
\dot{\phi} &= \frac{\kappa \cos(\phi)}{1 + \kappa \rho} - u_b, \quad (\rho, \phi) \in (0, +\infty) \times (-\pi/2, \pi/2)
\end{align*}
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\end{align*}
\] (16)

\[u_b = \frac{\kappa \cos(\phi)}{1 + \kappa \rho} - h'(\rho) \cos(\phi) + \mu \sin(\phi)\] (17)
Curve Tracking Dynamics

\[
\begin{align*}
\dot{\rho} &= -\sin(\phi) \\
\dot{\phi} &= \frac{\kappa \cos(\phi)}{1+\kappa \rho} - u_b, \quad (\rho, \phi) \in (0, +\infty) \times (-\pi/2, \pi/2) \\
u_b &= \frac{\kappa \cos(\phi)}{1+\kappa \rho} - h'(\rho) \cos(\phi) + \mu \sin(\phi)
\end{align*}
\]

\[
h(\rho) = \alpha \left\{ \rho + \frac{\rho_0^2}{\rho} - 2\rho_0 \right\}, \quad \rho_0 = \text{desired value for } \rho
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\begin{aligned}
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\end{aligned}
\]  

(16)

\[
u_b = \frac{\kappa \cos(\phi)}{1 + \kappa \rho} - h' (\rho) \cos(\phi) + \mu \sin(\phi)
\]

(17)

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h(\rho) = \alpha \left\{ \rho + \frac{\rho_0^2}{\rho} - 2\rho_0 \right\}, \quad \rho_0 = \text{desired value for } \rho
\]

(18)

\[
V(\rho, \phi) = -\ln \left( \cos(\phi) \right) + h(\rho)
\]

(19)
Curve Tracking Dynamics

\[
\begin{align*}
\dot{\rho} &= -\sin(\phi) \\
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\end{align*}
\]

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\[V(\rho, \phi) = -\ln(\cos(\phi)) + h(\rho)\]

\[U(\rho, \phi) = -h'(\rho) \sin(\phi) + \frac{1}{\mu} \int_0^{V(\rho, \phi)} \Gamma_0(m) dm\]
Robustly Forwardly Invariant Hexagons

We used $U$ to prove ISS results for the $(\rho - \rho_0, \phi)$ system, where

$$\dot{\rho} = -\sin(\phi), \quad \dot{\phi} = h'(\rho) \cos(\phi) - \mu \sin(\phi) + \delta,$$

and $\delta : [0, \infty) \to [-\delta^*, \delta^*]$, on certain forward invariant sets $H_i$.

View the state space $(0, \infty) \times (-\pi/2, \pi/2)$ of (21) as a union of compact hexagon shaped regions $H_1 \subseteq H_2 \subseteq \ldots \subseteq H_i \subseteq \ldots$.

For each $i$, all trajectories of (21) starting in $H_i$ for all $\delta : [0, \infty) \to [-\delta^*, \delta^*]$ stay in $H_i$.

Tight Disturbance Bound: Choose any $\delta^*_i \in (0, \min\{\Delta^*_i, \Delta^{**}_i\})$.

$$\Delta^*_i = \min\{|h'(\rho) \cos(\phi)| : (\rho, \phi) \in AB \cup ED\},$$

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and $\delta : [0, \infty) \rightarrow [-\delta_*, \delta_*]$, on certain forward invariant sets $H_i$. 

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\Delta_*^i = \min\{|h'(\rho) \cos(\phi)| : (\rho, \phi) \in AB \cup ED\}
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\] (21)
and $\delta : [0, \infty) \to [\delta_* i, \delta_* i],$ on certain forward invariant sets $H_i.$
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\]
and $\delta : [0, \infty) \rightarrow [-\delta^*_i, \delta^*_i]$, on certain forward invariant sets $H_i$.

View the state space $\mathbb{R}^2$ of (21) as a union of compact hexagon shaped regions $H_1 \subseteq H_2 \subseteq \ldots \subseteq H_i \subseteq \ldots$. 

[Diagram of hexagonal regions labeled A through F with a hexagonal set in the middle]
Robustly Forwardly Invariant Hexagons

We used $U$ to prove ISS results for the $(\rho - \rho_0, \phi)$ system, where
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(21)
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$$

$$
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$$
New Results (Mazenc, de Queiroz, M., ’11)

We solved the tracking and parameter identification problem for

\[
\begin{align*}
\dot{x} &= f(\xi) \\
\dot{z}_i &= g_i(\xi) + k_i(\xi)\theta_i + \psi_i u_i, \quad i = 1, 2, \ldots, s.
\end{align*}
\]

(5)

\[\xi = (x, z) \in \mathbb{R}^{r+s}. \quad (\theta, \psi) = (\theta_1, \ldots, \theta_s, \psi_1, \ldots, \psi_s) \in \mathbb{R}^{p_1 + \ldots + p_s + s}.\]

The \(C^2\) \(T\)-periodic reference trajectory \(\xi_R = (x_R, z_R)\) to be
tracked is assumed to satisfy \(\dot{x}_R(t) = f(\xi_R(t))\) \(\forall t \geq 0\).

Main PE Assumption: positive definiteness of the matrices

\[
\mathcal{M}_i \overset{\text{def}}{=} \int_0^T \lambda_i^\top(t)\lambda_i(t) \, dt \in \mathbb{R}^{(p_i+1) \times (p_i+1)},
\]

(6)

where \(\lambda_i(t) = (k_i(\xi_R(t)), \dot{z}_{R,i}(t) - g_i(\xi_R(t)))\) for \(i = 1, 2, \ldots, s\).
Adaptive Robust Curve Tracking

\[
\begin{align*}
\dot{\rho} &= -\sin(\phi) \\
\dot{\phi} &= \frac{\kappa \cos(\phi)}{1 + \kappa \rho} + K[u + \delta]
\end{align*}
\]

\[\xi = (\rho, \phi), \quad \theta_i = 0, \quad \psi_i = K, \quad f(\xi) = -\sin(\phi), \quad g_i(\xi) = \frac{\kappa \cos(\phi)}{1 + \kappa \rho}\]
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Take \(u = -u_b/\hat{K}\).
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\end{cases}
\]

\(\xi = (\rho, \phi), \theta_i = 0, \psi_i = K, f(\xi) = -\sin(\phi), g_i(\xi) = \frac{\kappa \cos(\phi)}{1 + \kappa \rho}\)

Take \(u = -u_b/\hat{K}\). We proved ISS for the dynamics

\[
\begin{cases}
\dot{\tilde{q}}_1 = -\sin(\tilde{q}_2) \\
\dot{\tilde{q}}_2 = \frac{\kappa \cos(\tilde{q}_2)}{1 + \kappa (\tilde{q}_1 + \rho_0)} - \frac{K}{\hat{K} + K} u_b + K\delta \\
\dot{\hat{K}} = -(\hat{K} + K - c_{\text{min}})(c_{\text{max}} - \hat{K} - K) \frac{\partial U}{\partial \phi} \frac{u_b}{\hat{K} + K}
\end{cases}
\]

for \((\tilde{q}_1, \tilde{q}_2, \hat{K}) = (\rho - \rho_0, \phi, \hat{K} - K)\) on each set in a nested sequence of hexagonal regions that fill the state space.
Summer 2011 Field Work at Grand Isle, LA

20 days of field work off Grand Isle. Search for oil spill remnants. Georgia Tech Savannah Robotics Team (led by Fumin Zhang).
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Conclusions

Nonlinear control systems are ubiquitous in aerospace, bio, electrical, and mechanical engineering. One central problem is to build functions called closed loop controllers that force desired tracking behaviors. We designed controllers for several applications including models with unknown parameters that we can identify. Our strict Lyapunov function approach gave key robustness properties such as input-to-state stability. We aim for extensions that cover input delays and state constraints that ensure collision avoidance.
Conclusions

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