Semiconcavity and optimal control: an intrinsic approach

Peter R. Wolenski

joint work with **Piermarco Cannarsa** and **Francesco Marino**

Louisiana State University

SADCO summer school, London

September 5-9, 2011

Outline

1. Optimal Control problems
   - Optimal control
   - Value functions and semiconcavity
   - Differential Inclusions (DI)

Outline

1. Optimal Control problems
   - Optimal control
   - Value functions and semiconcavity
   - Differential Inclusions (DI)
2. Smooth parameterizations?
   - Necessary conditions

Outline

1. Optimal Control problems
   - Optimal control
   - Value functions and semiconcavity
   - Differential Inclusions (DI)
2. Smooth parameterizations?
   - Necessary conditions
3. New (DI) assumptions
   - Examples
   - Consequences

Outline

1. Optimal Control problems
   - Optimal control
   - Value functions and semiconcavity
   - Differential Inclusions (DI)
2. Smooth parameterizations?
   - Necessary conditions
3. New (DI) assumptions
   - Examples
   - Consequences
4. New idea and results
   - A replacement for a priori estimates
   - Semiconcavity results with (DI)

Optimal control problems

Control Dynamics:

$$(\mathbf{CD}) \qquad \begin{cases} \dot{x}(s) = f\big(x(s), u(s)\big) & \text{a.e. } s \in [t, T] \\ u(s) \in U \quad \text{a.e. } s \in [t, T] \\ x(t) = x, \end{cases}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (x, u) and Lipschitz in x, the *admissible control* set $U \subseteq \mathbb{R}^m$ is compact, and $u : [t, T] \to \mathbb{R}^m$ is measurable.

Optimal control problems

Control Dynamics:

$$(\mathbf{CD}) \qquad \begin{cases} \dot{x}(s) = f\big(x(s), u(s)\big) & \text{a.e. } s \in [t, T] \\ u(s) \in U & \text{a.e. } s \in [t, T] \\ x(t) = x, \end{cases}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (x, u) and Lipschitz in x, the *admissible control* set $U \subseteq \mathbb{R}^m$ is compact, and $u : [t, T] \to \mathbb{R}^m$ is measurable.

Two classic problems:

1. **MinTime:** Given a closed *target* set $S \subseteq \mathbb{R}^n$, the problem is

   $\min\ (T - t)$ over $\big(x(\cdot), u(\cdot)\big)$ satisfying **(CD)** and $x(T) \in S$.

   The optimal value $T(x)$ is the **minimum time function**.

Optimal control problems

Control Dynamics:

$$\textbf{(CD)} \qquad \begin{cases} \dot{x}(s) = f\big(x(s), u(s)\big) & \text{a.e. } s \in [t, T] \\ u(s) \in U \quad \text{a.e. } s \in [t, T] \\ x(t) = x, \end{cases}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (x, u) and Lipschitz in x, the *admissible control* set $U \subseteq \mathbb{R}^m$ is compact, and $u : [t, T] \to \mathbb{R}^m$ is measurable.

Two classic problems:

1. **MinTime:** Given a closed *target* set $S \subseteq \mathbb{R}^n$, the problem is

$$\min\ (T - t) \quad \text{over } \big(x(\cdot), u(\cdot)\big) \text{ satisfying } \textbf{(CD)} \text{ and } x(T) \in S.$$

The optimal value $T(x)$ is the **minimum time function**.

2. **Mayer problem:** Given *endpoint cost* $\ell : \mathbb{R}^n \to \mathbb{R}$, the problem is

$$\min\ \ell\big(x(T)\big) \quad \text{over } \big(x(\cdot), u(\cdot)\big) \text{ satisfying } \textbf{(CD)}.$$

The optimal value $V(t, x)$ is the **value function**.

Regularity of value functions - SemiConCavity **(SCC)**

A natural regularity property for $T(\cdot)$ and $V(\cdot,\cdot)$ is the property of being **semiconcave**. A locally Lipschitz function $g : \mathbb{R}^m \to \mathbb{R}$ is semiconcave provided there exists $k > 0$ so that

$$\frac{1}{2}\left[g(x+z) + g(x-z)\right] - g(x) \leq k \, \|z\|^2 \quad \forall x, z \in \mathbb{R}^n.$$

(the three point property)

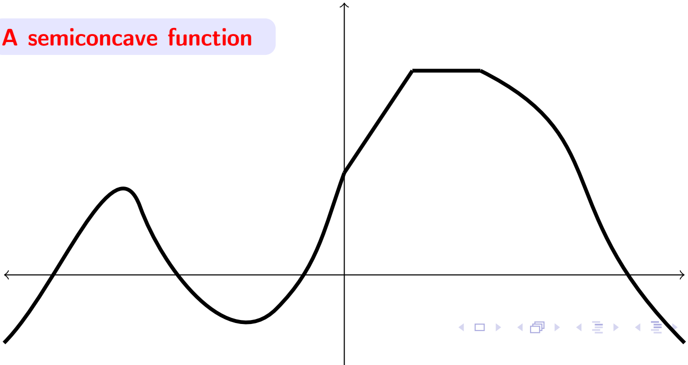
Regularity of value functions - SemiConCavity **(SCC)**

A natural regularity property for $T(\cdot)$ and $V(\cdot,\cdot)$ is the property of being **semiconcave**. A locally Lipschitz function $g : \mathbb{R}^m \to \mathbb{R}$ is semiconcave provided there exists $k > 0$ so that

$$\frac{1}{2}\left[g(x+z) + g(x-z)\right] - g(x) \leq k\, \|z\|^2 \quad \forall x, z \in \mathbb{R}^n.$$

(the three point property)

A semiconcave function



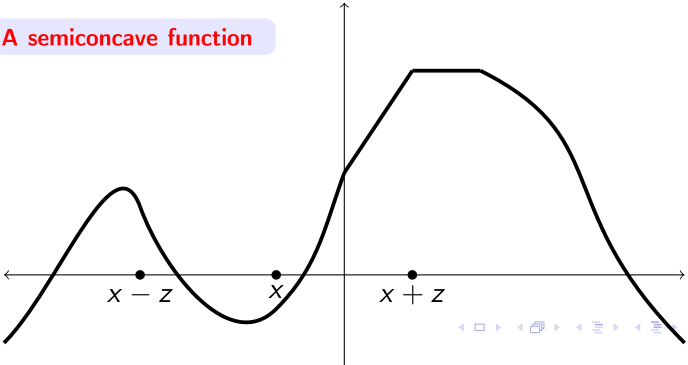
Regularity of value functions - SemiConCavity **(SCC)**

A natural regularity property for $T(\cdot)$ and $V(\cdot,\cdot)$ is the property of being **semiconcave**. A locally Lipschitz function $g : \mathbb{R}^m \to \mathbb{R}$ is semiconcave provided there exists $k > 0$ so that

$$\frac{1}{2}\left[g(x+z) + g(x-z)\right] - g(x) \leq k\,\|z\|^2 \quad \forall x,\, z \in \mathbb{R}^n.$$

(the three point property)

A semiconcave function

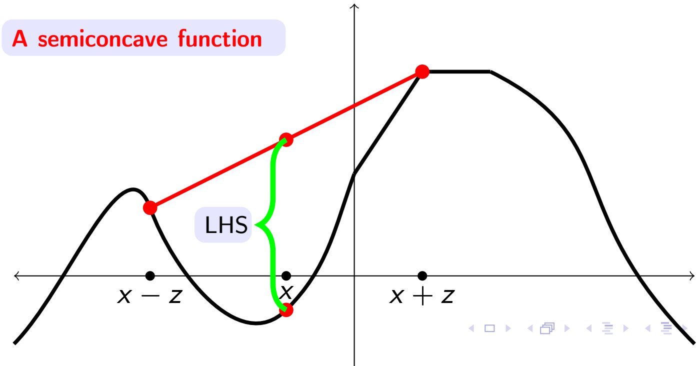


Regularity of value functions - SemiConCavity **(SCC)**

A natural regularity property for $T(\cdot)$ and $V(\cdot,\cdot)$ is the property of being **semiconcave**. A locally Lipschitz function $g:\mathbb{R}^m \to \mathbb{R}$ is semiconcave provided there exists $k>0$ so that

$$\frac{1}{2}\left[g(x+z)+g(x-z)\right]-g(x) \le k\,\|z\|^2 \quad \forall x,\, z \in \mathbb{R}^n.$$

(the three point property)



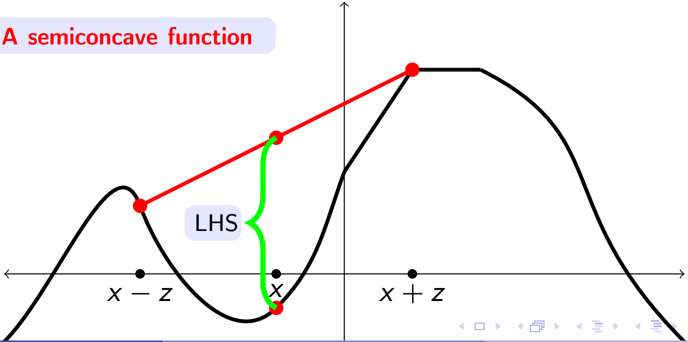
Regularity of value functions - SemiConCavity **(SCC)**

A natural regularity property for $T(\cdot)$ and $V(\cdot,\cdot)$ is the property of being **semiconcave**. A locally Lipschitz function $g : \mathbb{R}^m \to \mathbb{R}$ is semiconcave provided there exists $k > 0$ so that

$$\frac{1}{2}\left[g(x+z) + g(x-z)\right] - g(x) \leq k\, \|z\|^2 \quad \forall x, z \in \mathbb{R}^n.$$

(the three point property)

A semiconcave function



A geometric description of SCC

A Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ is (SCC) if and only if $\exists\, \sigma > 0$ with

$$\begin{aligned} g(x) &= \inf\{q(x) : q(x) = \sigma x^2 + bx + c,\ g(x) \le q(x)\} \\ &= \inf\{q(x,\alpha) : \alpha \in \mathcal{A}\}, \end{aligned}$$

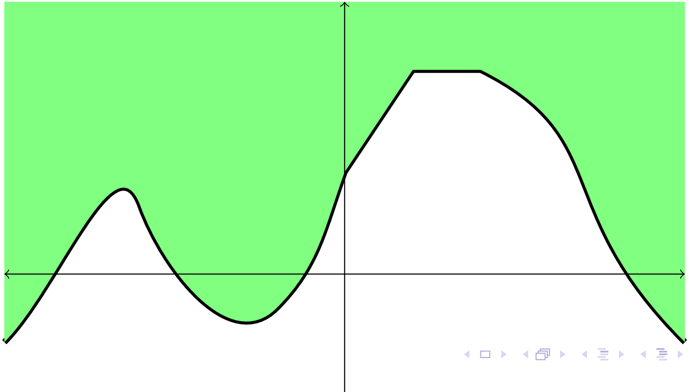
where $(x,\alpha) \mapsto q(x,\alpha)$ is C^{1+} in x and continuous in (x,α).

A geometric description of SCC

A Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ is (SCC) if and only if $\exists\, \sigma > 0$ with

$$\begin{aligned} g(x) &= \inf\big\{q(x) : q(x) = \sigma x^2 + bx + c,\ g(x) \le q(x)\big\} \\ &= \inf\big\{q(x,\alpha) : \alpha \in \mathcal{A}\big\}, \end{aligned}$$

where $(x,\alpha) \mapsto q(x,\alpha)$ is C^{1+} in x and continuous in (x,α).

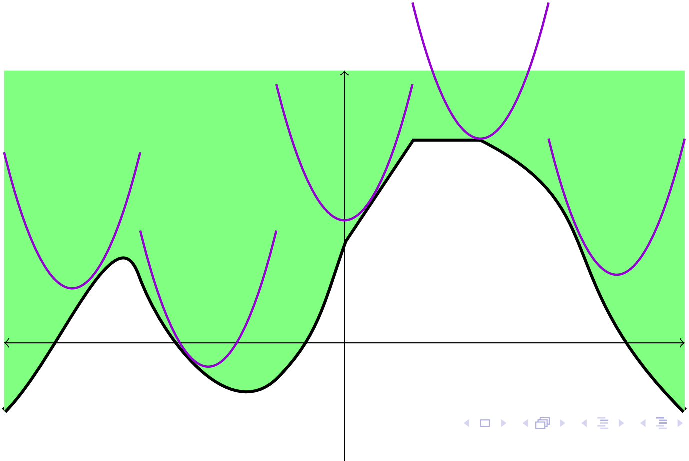


A geometric description of SCC

A Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ is (SCC) if and only if $\exists\, \sigma > 0$ with

$$\begin{aligned} g(x) &= \inf\{q(x) : q(x) = \sigma x^2 + bx + c,\ g(x) \leq q(x)\} \\ &= \inf\{q(x, \alpha) : \alpha \in \mathcal{A}\}, \end{aligned}$$

where $(x, \alpha) \mapsto q(x, \alpha)$ is C^{1+} in x and continuous in (x, α).

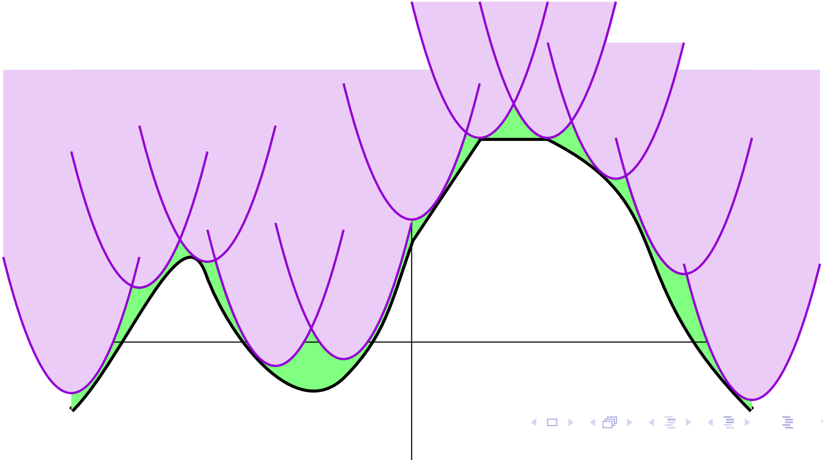


A geometric description of SCC

A Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ is (SCC) if and only if $\exists\, \sigma > 0$ with

$$\begin{aligned} g(x) &= \inf\{q(x) : q(x) = \sigma x^2 + bx + c,\ g(x) \le q(x)\} \\ &= \inf\{q(x,\alpha) : \alpha \in \mathcal{A}\}, \end{aligned}$$

where $(x,\alpha) \mapsto q(x,\alpha)$ is C^{1+} in x and continuous in (x,α).

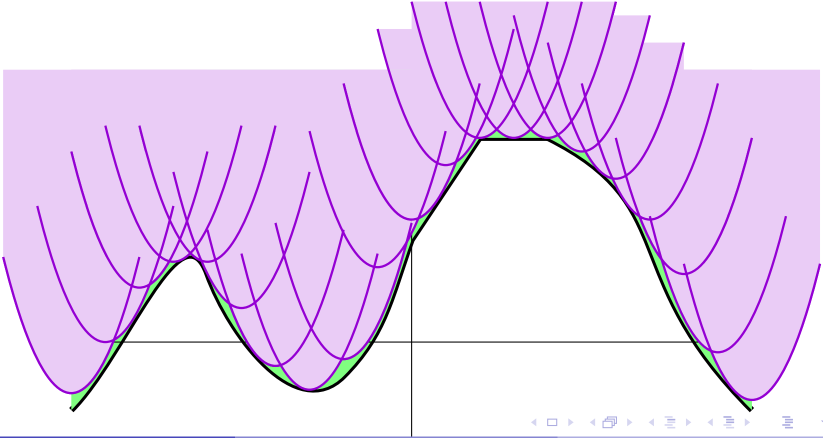


A geometric description of SCC

A Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ is (SCC) if and only if $\exists\, \sigma > 0$ with

$$\begin{aligned} g(x) &= \inf\{q(x) : q(x) = \sigma x^2 + bx + c,\ g(x) \le q(x)\} \\ &= \inf\{q(x, \alpha) : \alpha \in \mathcal{A}\}, \end{aligned}$$

where $(x, \alpha) \mapsto q(x, \alpha)$ is C^{1+} in x and continuous in (x, α).



Previous results yielding (SCC)

There is considerable literature on value functions being (SCC). Most relevant here: Cannarsa, Frankowska, Sinestrari, McEneaney.

Previous results yielding (SCC)

There is considerable literature on value functions being (SCC). Most relevant here: Cannarsa, Frankowska, Sinestrari, McEneaney.

Basic idea with (CD): (Illustration with Min Time)

We seek an upper bound (by $k\|z\|^2$) of

$$T(x+z)+T(x-z)-2T(x).$$

Take an optimal solution starting from x and use it to construct feasible solutions from $x \pm z$ that will yield the appropriate estimates.

Previous results yielding (SCC)

There is considerable literature on value functions being (SCC). Most relevant here: Cannarsa, Frankowska, Sinestrari, McEneaney.

Basic idea with (CD): (Illustration with Min Time)

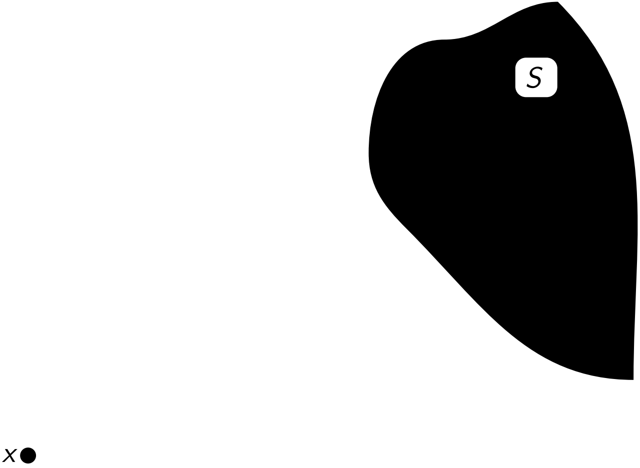
We seek an upper bound (by $k\|z\|^2$) of

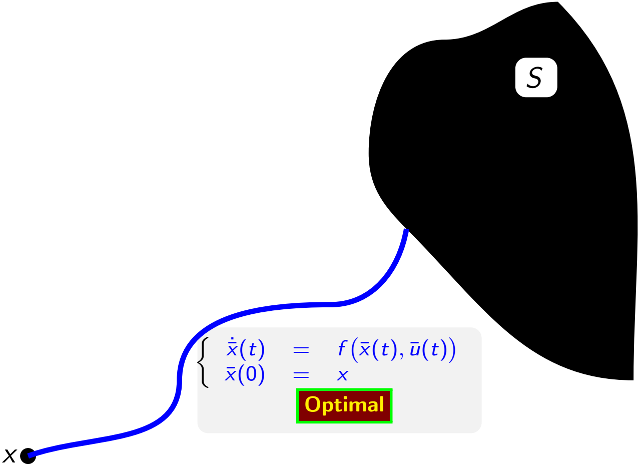
$$T(x+z)+T(x-z)-2T(x).$$

Take an optimal solution starting from x and use it to construct feasible solutions from $x \pm z$ that will yield the appropriate estimates.

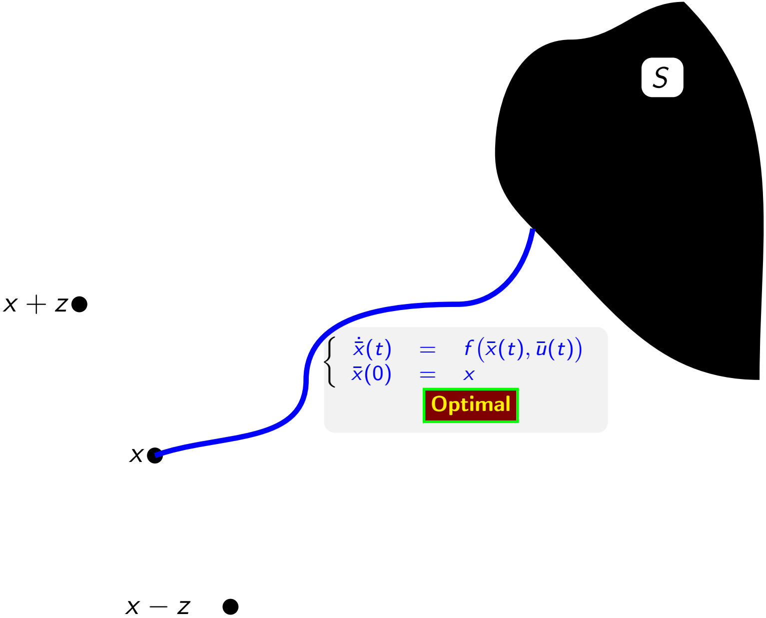
Assume that $x \mapsto f(x,u)$ is C^{1+} and take $\left(\bar{x}(\cdot), \bar{u}(\cdot)\right)$ optimal. Use a priori estimates on the ODEs

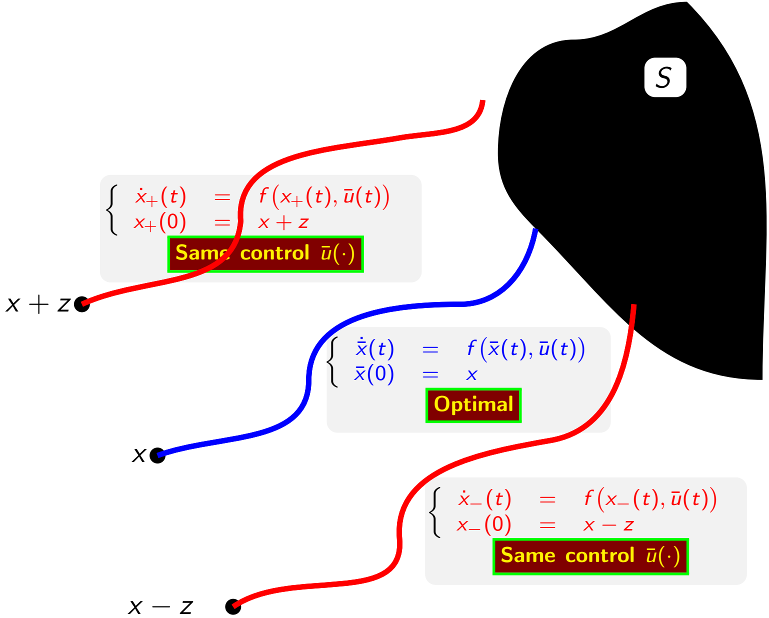
$$(\mathbf{ODE}_{\pm}) \qquad \begin{cases} \dot{x}_{\pm}(s) = f\left(x_{\pm}(s), \bar{u}(s)\right) \text{ a.e. } s \in [t,T] \\ x_{\pm}(t) = x \pm z. \end{cases}$$


S
x



$$\begin{cases} \dot{\bar{x}}(t) & = & f\left(\bar{x}(t), \bar{u}(t)\right) \\ \bar{x}(0) & = & x \end{cases}$$


S
$x + z$
x
$x - z$
$\begin{cases} \dot{\bar{x}}(t) = f\left(\bar{x}(t), \bar{u}(t)\right) \\ \bar{x}(0) = x \end{cases}$
Optimal


S
$\begin{cases} \dot{x}_+(t) = f(x_+(t), \bar{u}(t)) \\ x_+(0) = x + z \end{cases}$
Same control $\bar{u}(\cdot)$
$x + z$
$\begin{cases} \dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t)) \\ \bar{x}(0) = x \end{cases}$
Optimal
x
$\begin{cases} \dot{x}_-(t) = f(x_-(t), \bar{u}(t)) \\ x_-(0) = x - z \end{cases}$
Same control $\bar{u}(\cdot)$
$x - z$

This proof depends on the parameterization

A priori estimates rely on the specific parameterization $(x, u) \mapsto f(x, u)$ of the dynamics, but the value functions $T(\cdot)$ and $V(\cdot, \cdot)$ do not.

This proof depends on the parameterization

A priori estimates rely on the specific parameterization $(x, u) \mapsto f(x, u)$ of the dynamics, but the value functions $T(\cdot)$ and $V(\cdot, \cdot)$ do not.

Important:

To obtain second order (SCC) estimates, one definitely requires more than mere Lipschitz of the map $x \mapsto f(x, u)$.

This proof depends on the parameterization

A priori estimates rely on the specific parameterization $(x, u) \mapsto f(x, u)$ of the dynamics, but the value functions $T(\cdot)$ and $V(\cdot, \cdot)$ do not.

Important:

To obtain second order (SCC) estimates, one definitely requires more than mere Lipschitz of the map $x \mapsto f(x, u)$.

The question is **What?**

This proof depends on the parameterization

A priori estimates rely on the specific parameterization $(x,u) \mapsto f(x,u)$ of the dynamics, but the value functions $T(\cdot)$ and $V(\cdot,\cdot)$ do not.

Important:

To obtain second order (SCC) estimates, one definitely requires more than mere Lipschitz of the map $x \mapsto f(x,u)$.

The question is **What?**

Recall the previous work assumed $x \mapsto f(x,u)$ is C^{1+}.

(Very) simple example

Note that $T(\cdot)$ and $V(\cdot,\cdot)$ depend only on the trajectories $x(\cdot)$ and **NOT** in the parameterization of the *admissible velocity set*:

$$F(x) := \left\{ f(x,u) : u \in U \right\}.$$

(Very) simple example

Note that $T(\cdot)$ and $V(\cdot,\cdot)$ depend only on the trajectories $x(\cdot)$ and **NOT** in the parameterization of the *admissible velocity set*:

$$F(x) := \{f(x,u) : u \in U\}.$$

Note the admissible velocity multifunction $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by $F(x) = \Big[-|x|, |x|\Big]$ can be parameterized two ways:

$$F(x) = \begin{cases} \{x \cdot u & : |u| \leq 1\} \\ \{|x| \cdot u & : |u| \leq 1\} \end{cases}$$

The former is smoothly parameterized whereas the latter is not.

(Very) simple example

Note that $T(\cdot)$ and $V(\cdot,\cdot)$ depend only on the trajectories $x(\cdot)$ and **NOT** in the parameterization of the *admissible velocity set*:

$$F(x) := \{f(x,u) : u \in U\}.$$

Note the admissible velocity multifunction $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by $F(x) = \Big[-|x|, |x|\Big]$ can be parameterized two ways:

$$F(x) = \begin{cases} \{x \cdot u \quad : |u| \leq 1\} \\ \{|x| \cdot u \quad : |u| \leq 1\} \end{cases}$$

The former is smoothly parameterized whereas the latter is not.

Trajectories coincide, but theorems only apply to the former.

Differential Inclusions and Filippov's Lemma

The set of solutions to the Differential Inclusion

$$\textbf{(DI)} \qquad \begin{cases} \dot{x}(s) \in F\big(x(s)\big) \text{ a.e. } s \in [t, T] \\ x(t) = x \end{cases}$$

does not depend on the particular parameterization. This is (essentially) the content of the well-known Filippov's Lemma.

Differential Inclusions and Filippov's Lemma

The set of solutions to the Differential Inclusion

$$\textbf{(DI)} \qquad \begin{cases} \dot{x}(s) \in F\big(x(s)\big) \text{ a.e. } s \in [t, T] \\ x(t) = x \end{cases}$$

does not depend on the particular parameterization. This is (essentially) the content of the well-known Filippov's Lemma.

Natural question:

For which F is the value function semiconcave?

Differential Inclusions and Filippov's Lemma

The set of solutions to the Differential Inclusion

$$\textbf{(DI)} \qquad \begin{cases} \dot{x}(s) \in F\big(x(s)\big) \text{ a.e. } s \in [t, T] \\ x(t) = x \end{cases}$$

does not depend on the particular parameterization. This is (essentially) the content of the well-known Filippov's Lemma.

Natural question:

For which F is the value function semiconcave?

A satisfactory answer should be given in terms of F, or equivalently, by the **Hamiltonian** $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by:

$$H(x,p) = \sup_{v \in F(x)} \big\langle v, p \big\rangle.$$

Equivalence of F and H

We assume throughout that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies the following Standard Hypotheses:

$$(SH)_+ \begin{cases} \mathbf{1)}\ F(x) \text{ is nonempty, convex, and compact } \forall\, x, \\ \mathbf{2)}\ F \text{ is Lipschitz on bounded sets w.r.t. Hausdorff metric;} \\ \mathbf{3)}\ \exists\, r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \leq r(1 + |x|). \end{cases}$$

Equivalence of F and H

We assume throughout that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies the following Standard Hypotheses:

$$(SH)_+ \begin{cases} \textbf{1)}\ F(x) \text{ is nonempty, convex, and compact } \forall\, x, \\ \textbf{2)}\ F \text{ is Lipschitz on bounded sets w.r.t. Hausdorff metric;} \\ \textbf{3)}\ \exists\, r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \leq r(1 + |x|). \end{cases}$$

Such assumptions on F give way to equivalent conditions on H because

$$v \in F(x) \quad \Longleftrightarrow \quad \langle v, p\rangle \leq H(x, p) \quad \forall p \in \mathbb{R}^n$$

Equivalence of F and H

We assume throughout that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies the following Standard Hypotheses:

$$(SH)_+ \begin{cases} \textbf{1)}\; F(x) \text{ is nonempty, convex, and compact } \forall\, x, \\ \textbf{2)}\; F \text{ is Lipschitz on bounded sets w.r.t. Hausdorff metric;} \\ \textbf{3)}\; \exists\, r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \leq r(1 + |x|). \end{cases}$$

Such assumptions on F give way to equivalent conditions on H because

$$v \in F(x) \quad \Longleftrightarrow \quad \langle v, p \rangle \leq H(x, p) \quad \forall p \in \mathbb{R}^n$$

$$(SH)_+ \begin{cases} \textbf{1)}\; \forall\, x \in \mathbb{R}^n, H(x,p) \text{ is finite and convex,} \\ \qquad\qquad \text{positively homogeneous in } p; \\ \textbf{2)}\; \forall M > 0, \exists k > 0 \text{ so that } \forall \|x\|, \|y\| \leq M, p \in \mathbb{R}^n, \\ \qquad\qquad \big| H(x,p) - H(y,p) \big| \leq k\, \|p\|\, \|x - y\|; \\ \textbf{3)}\; \exists\, r > 0 \text{ so that } H(x,p) \leq r\, \|p\|\, (1 + |x|) \quad \forall\, x, p \in \mathbb{R}^n. \end{cases}$$

Smooth parameterizations?

Perhaps one can characterize those multifunctions that have a smooth parameterization:

Question:

Given $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, when does there exist $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ that is C^1 in the first coordinate and satisfies

$$F(x) := \left\{ f(x,u) : u \in U \right\}?$$

Smooth parameterizations?

Perhaps one can characterize those multifunctions that have a smooth parameterization:

Question:

Given $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, when does there exist $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ that is C^1 in the first coordinate and satisfies

$$F(x) := \left\{ f(x,u) : u \in U \right\}?$$

This seems virtually impossible to answer. **Worse:** Even sufficient conditions for smooth *selections* seems intractable:

A simpler (?) question:

Under what conditions on F does there exist a C^1 function $f : \mathbb{R}^n \to \mathbb{R}^n$ so that $f(x) \in F(x) \quad \forall x$?

Necessary conditions for smooth parameterizations

We can say when a smooth parameterization does **NOT** exist!

Necessary conditions for smooth parameterizations

We can say when a smooth parameterization does **NOT** exist!

Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is smoothly parameterized and H has the form

$$H(x,p) = \sup\Big\{ \langle f(x,u), p \rangle \ : \ u \in U \Big\},$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ has f and $\frac{\partial f}{\partial x}$ both continuous in (x,u).

Necessary conditions for smooth parameterizations

We can say when a smooth parameterization does **NOT** exist!

Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is smoothly parameterized and H has the form

$$H(x,p) = \sup\left\{ \langle f(x,u), p \rangle \ : \ u \in U \right\},$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ has f and $\frac{\partial f}{\partial x}$ both continuous in (x,u).

Then for $0 \neq p \in \mathbb{R}^n$, we have

(H1) The map $x \mapsto H(x,p)$ is semiconvex;

Necessary conditions for smooth parameterizations

We can say when a smooth parameterization does **NOT** exist!

Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is smoothly parameterized and H has the form

$$H(x,p) = \sup\Big\{ \langle f(x,u), p \rangle \ : \ u \in U \Big\},$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ has f and $\frac{\partial f}{\partial x}$ both continuous in (x,u).

Then for $0 \neq p \in \mathbb{R}^n$, we have

(H1) The map $x \mapsto H(x,p)$ is semiconvex; and

(NC) If $H(x,p) = -H(x,-p)$, then

$$\partial_x H(x,p) = -\partial_x H(x,-p).$$

Necessary conditions for smooth parameterizations

We can say when a smooth parameterization does **NOT** exist!

Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is smoothly parameterized and H has the form

$$H(x,p) = \sup\Big\{ \langle f(x,u), p \rangle \ : \ u \in U \Big\},$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ has f and $\frac{\partial f}{\partial x}$ both continuous in (x,u).

Then for $0 \neq p \in \mathbb{R}^n$, we have

(H1) The map $x \mapsto H(x,p)$ is semiconvex; and

(NC) If $H(x,p) = -H(x,-p)$, then

$$\partial_x H(x,p) = -\partial_x H(x,-p).$$

If the assumption of a smooth parameterization is replaced by the existence of a smooth selection, then the conclusion of **(NC)** is

$$\partial_x H(x,p) \bigcap -\partial_x H(x,-p) \neq \emptyset.$$

Illustration of (NC) with $n = 1$

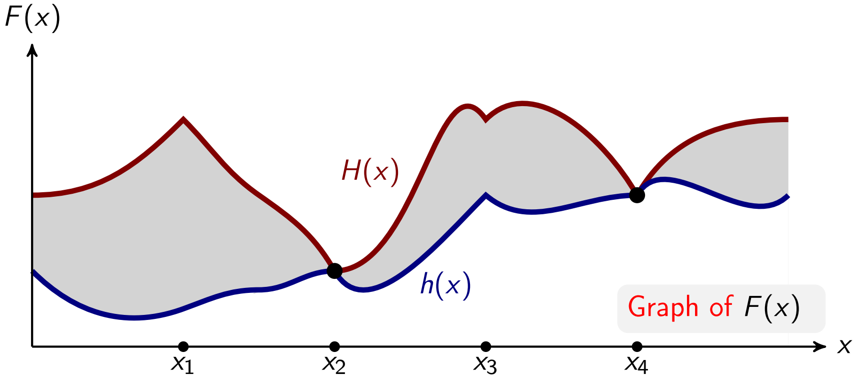


Illustration of (NC) with $n = 1$

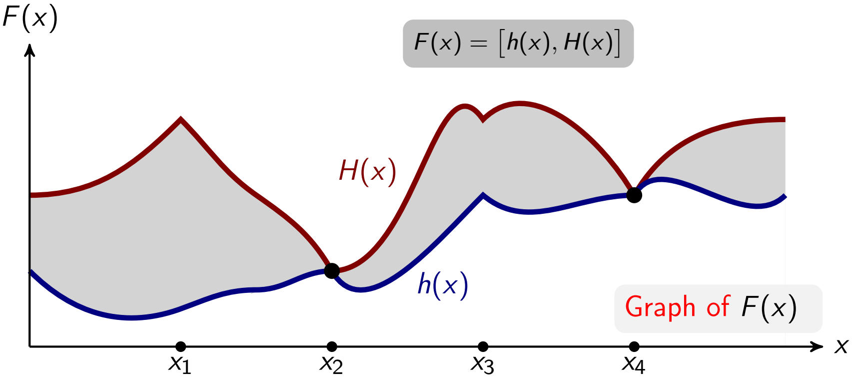


Illustration of (NC) with $n = 1$

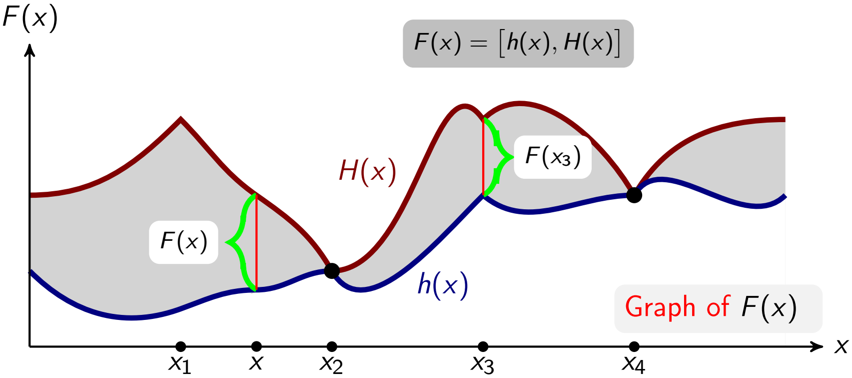
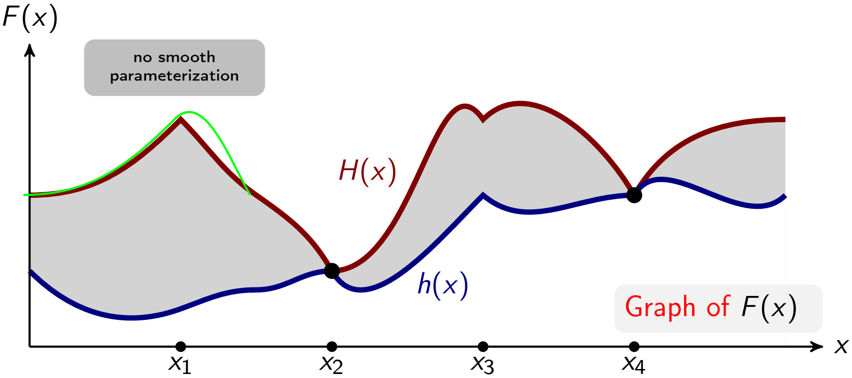
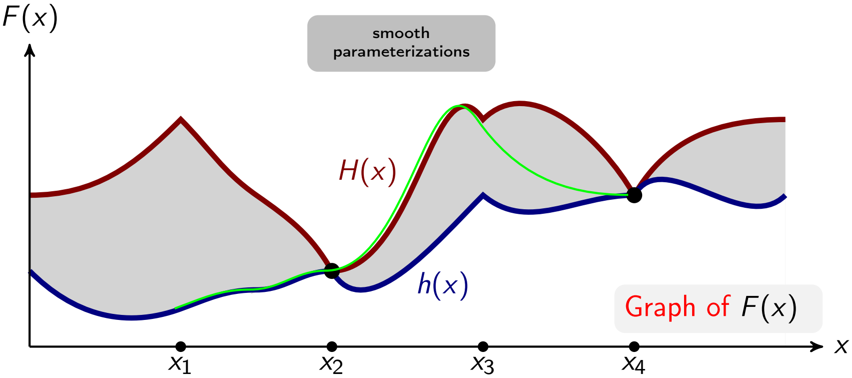


Illustration of (NC) with $n = 1$



x_1: No smooth parameterization since $x \mapsto H(x)$ not semiconvex.

Illustration of (NC) with $n = 1$



x_2, x_3: Smooth parameterizations are possible between x_1 and x_4.

Illustration of (NC) with $n = 1$

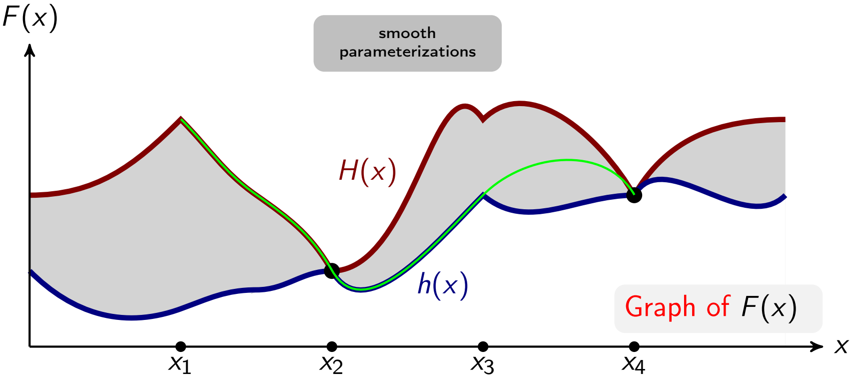


Illustration of (NC) with $n = 1$

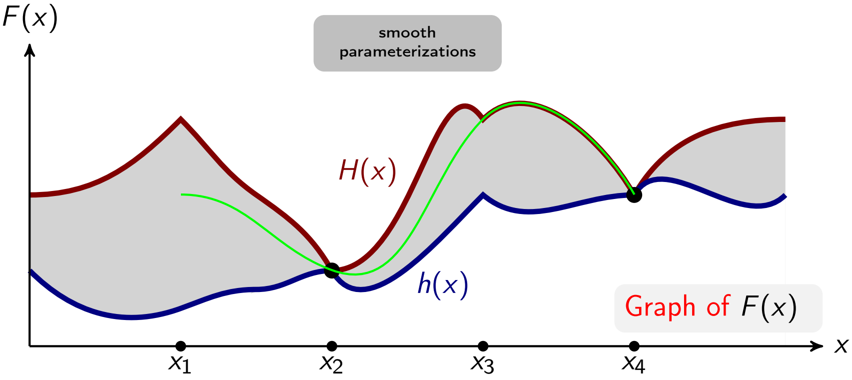


Illustration of (NC) with $n = 1$

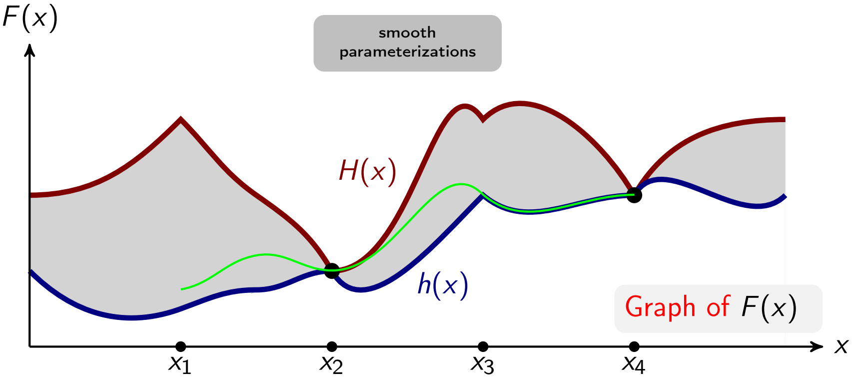
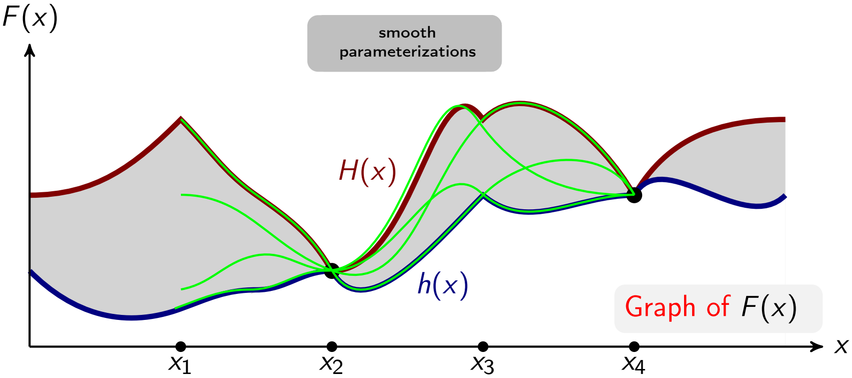
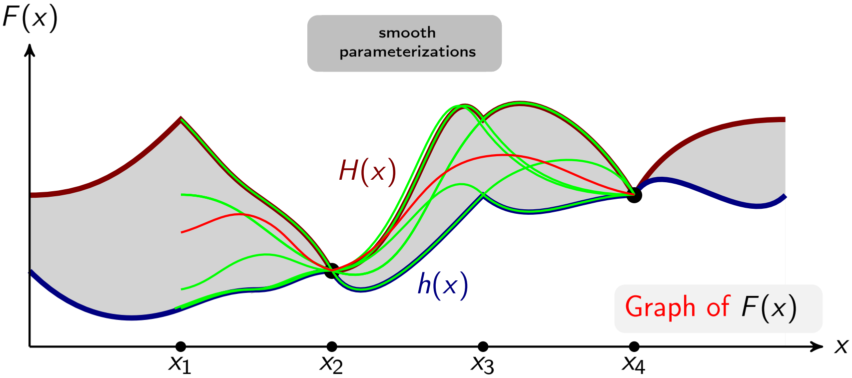


Illustration of (NC) with $n = 1$



x_2, x_3: Smooth parameterizations are possible between x_1 and x_4.

Illustration of (NC) with $n = 1$



x_2, x_3: Smooth parameterizations are possible between x_1 and x_4.

Illustration of (NC) with $n = 1$

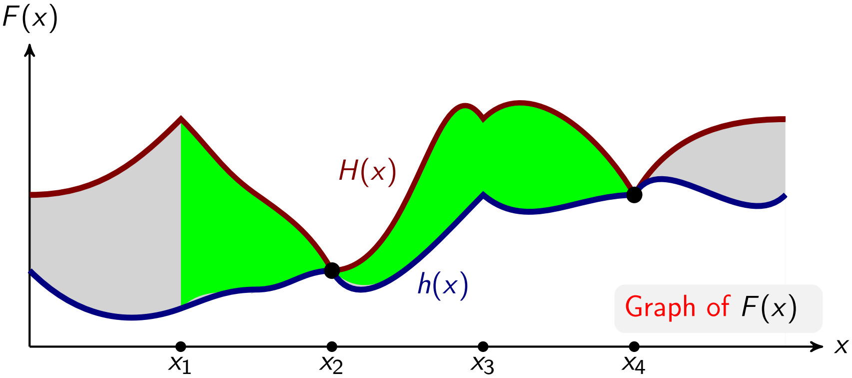
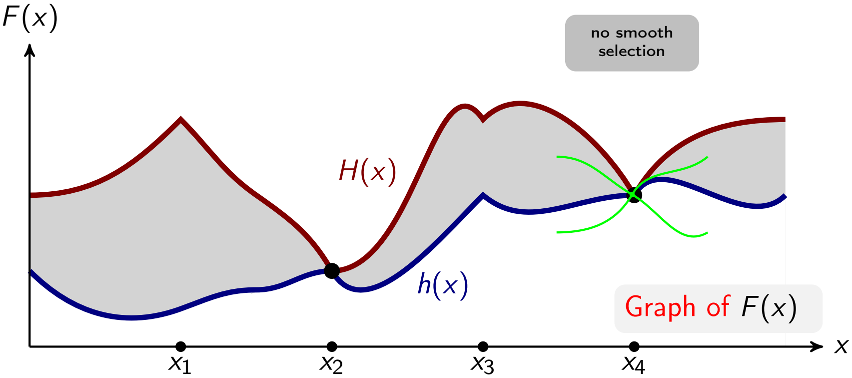


Illustration of (NC) with $n = 1$



x_4: No smooth selection.

Proof of **(NC)**

Proof.

Note that the assumption

$$H(x,p) = -H(x,-p)$$

means that

$$\sup_{u\in U}\langle f(x,u),p\rangle = -\sup_{v\in F(x)}\langle v,-p\rangle = \inf_{v\in F(x)}\langle v,p\rangle = \inf_{u\in U}\langle f(x,u),p\rangle$$

That is, the assumption is that every $u \in U$ both minimizes and maximizes the quantity $\langle f(x,u),p\rangle$.

Proof of **(NC)**

Proof.

Note that the assumption

$$\boxed{H(x,p) = -H(x,-p)}$$

means that

$$\sup_{u\in U}\langle f(x,u),p\rangle = -\sup_{v\in F(x)}\langle v,-p\rangle = \inf_{v\in F(x)}\langle v,p\rangle = \inf_{u\in U}\langle f(x,u),p\rangle$$

That is, the assumption is that every $u \in U$ both minimizes and maximizes the quantity $\langle f(x,u),p\rangle$. By a theorem on *nonsmooth differentiation* of max functions, one has

$$\partial_x H(x,p) = \overline{\text{co}}\left\{\nabla_x f(x,u)\,p \;:\; u\in U\right\}$$

and

$$\partial_x H(x,-p) = \overline{\text{co}}\left\{-\nabla_x f(x,u)\,p \;:\; u\in U\right\},$$

from which **(NC)** follows:

Proof of **(NC)**

Proof.

Note that the assumption

$$H(x,p) = -H(x,-p)$$

means that

$$\sup_{u\in U}\langle f(x,u),p\rangle = -\sup_{v\in F(x)}\langle v,-p\rangle = \inf_{v\in F(x)}\langle v,p\rangle = \inf_{u\in U}\langle f(x,u),p\rangle$$

That is, the assumption is that every $u \in U$ both minimizes and maximizes the quantity $\langle f(x,u),p\rangle$. By a theorem on *nonsmooth differentiation* of max functions, one has

$$\partial_x H(x,p) = \overline{\text{co}}\left\{\nabla_x f(x,u)\,p \;:\; u \in U\right\}$$

and

$$\partial_x H(x,-p) = \overline{\text{co}}\left\{-\nabla_x f(x,u)\,p \;:\; u \in U\right\},$$

from which **(NC)** follows: $\partial_x H(x,p) = -\partial_x H(x,-p)$. □

New DI assumptions

We abandon looking for smooth parameterizations, and introduce:

$$
\textbf{(H)} \quad \begin{cases} \textbf{1)} & x \mapsto H(x,p) \text{ is semiconvex, and} \\ \textbf{2)} & \text{The gradient } \nabla_p H(x,p) \text{ exists and is locally Lipschitz in } x. \end{cases}
$$

New DI assumptions

We abandon looking for smooth parameterizations, and introduce:

$$(\mathbf{H}) \quad \begin{cases} \mathbf{1)} & x \mapsto H(x,p) \text{ is semiconvex, and} \\ \mathbf{2)} & \text{The gradient } \nabla_p H(x,p) \text{ exists and is locally Lipschitz in } x. \end{cases}$$

Class of examples:

One can generate a class of examples that satisfy **(H)** but do not satisfy **(NC)**, and therefore could not have a C^1 parameterization:

New DI assumptions

We abandon looking for smooth parameterizations, and introduce:

$$(\mathbf{H}) \quad \begin{cases} \mathbf{1)} & x \mapsto H(x,p) \text{ is semiconvex, and} \\ \mathbf{2)} & \text{The gradient } \nabla_p H(x,p) \text{ exists and is locally Lipschitz in } x. \end{cases}$$

Class of examples:

One can generate a class of examples that satisfy **(H)** but do not satisfy **(NC)**, and therefore could not have a C^1 parameterization:

$n = 1$: Let $F(x) := \left[h(x), H(x)\right]$ where $-h(\cdot)$ and $H(\cdot)$ are semiconvex. These always satisfy **(H)**, and could satisfy **(NC)** only if $h(x) = H(x) \Rightarrow \partial h(x) = \partial H(x)$.

New DI assumptions

We abandon looking for smooth parameterizations, and introduce:

$$(\mathbf{H}) \quad \begin{cases} \mathbf{1)} & x \mapsto H(x,p) \text{ is semiconvex, and} \\ \mathbf{2)} & \text{The gradient } \nabla_p H(x,p) \text{ exists and is locally Lipschitz in } x. \end{cases}$$

Class of examples:

One can generate a class of examples that satisfy **(H)** but do not satisfy **(NC)**, and therefore could not have a C^1 parameterization:

$n = 1$: Let $F(x) := \left[h(x), H(x)\right]$ where $-h(\cdot)$ and $H(\cdot)$ are semiconvex. These always satisfy **(H)**, and could satisfy **(NC)** only if $h(x) = H(x) \Rightarrow \partial h(x) = \partial H(x)$.

$n > 1$: Let $F(x) := f(x) + r(x)\overline{\mathbb{B}}$ where $f(\cdot)$ is C^2 and $r : \mathbb{R}^n \to [0, \infty)$ is semiconvex. Then $H(x,p) = \langle f(x), p\rangle + r(x)\|p\|$, and so **(H)** is satisfied. Then **(NC)** is satisfied only if $r(x) = 0$ implies $\partial r(x) = -\partial r(x)$.

Consequences, part I

Consequences of (H1):

The semiconvexity of $x \mapsto H(x,p)$ implies

$$\partial_{x,p} H(x,p) \subseteq \partial_x H\big(x,p\big) \times \partial_p H\big(x,p\big).$$

Consequences, part I

Consequences of (H1):

The semiconvexity of $x \mapsto H(x,p)$ implies

$$\partial_{x,p} H(x,p) \subseteq \partial_x H\big(x,p\big) \times \partial_p H\big(x,p\big).$$

The significance of this result is in utilizing a nonsmooth maximum principle (Clarke 1975):

Suppose $\bar{x}(\cdot)$ is optimal in one of the classical problems with **(DI)** dynamics. Then there exists an adjoint arc $\bar{p}(\cdot)$ for which

$$\big(-\dot{\bar{p}}(s), \dot{\bar{x}}(s)\big) \in \partial_{x,p} H\big(\bar{x}(s), \bar{p}(s)\big)$$

(plus transversality conditions).

Consequences, part I

Consequences of (H1):

The semiconvexity of $x \mapsto H(x,p)$ implies

$$\partial_{x,p} H(x,p) \subseteq \partial_x H\big(x,p\big) \times \partial_p H\big(x,p\big).$$

The significance of this result is in utilizing a nonsmooth maximum principle (Clarke 1975):

Suppose $\bar{x}(\cdot)$ is optimal in one of the classical problems with **(DI)** dynamics. Then there exists an adjoint arc $\bar{p}(\cdot)$ for which

$$\big(-\dot{\bar{p}}(s), \dot{\bar{x}}(s)\big) \in \partial_{x,p} H\big(\bar{x}(s), \bar{p}(s)\big)$$

(plus transversality conditions). Thus the dynamics of a Hamiltonian arc $\big(\bar{x}(\cdot), \bar{p}(\cdot)\big)$ "splits" into a much more usable form:

$$-\dot{\bar{p}}(s) \in \partial_x H\big(\bar{x}(s), \bar{p}(s)\big) \quad \text{and} \quad \dot{\bar{x}}(s) \in \partial_p H\big(\bar{x}(s), \bar{p}(s)\big)$$

Consequences, part II

Consequences of (H2):

That the gradient $\nabla_p H(x,p)$ exists means that the argmax of $\sup_{v \in F(x)} \langle v, p \rangle$ is unique - we denote it by $f_p(x) \in F(x)$.

Consequences, part II

Consequences of (H2):

That the gradient $\nabla_p H(x,p)$ exists means that the argmax of $\sup_{v \in F(x)} \langle v, p \rangle$ is unique - we denote it by $f_p(x) \in F(x)$.

We have also assumed $x \mapsto f_p(x)$ is Lipschitz . If $p(\cdot)$ is any continuous function, then the ODE

$$(\mathbf{ODE})_x \qquad \begin{cases} \dot{x}(t) = f_{p(t)}\big(x(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x, \end{cases}$$

satisfies standard Carathéodory-type assumptions

Consequences, part II

Consequences of (H2):

That the gradient $\nabla_p H(x,p)$ exists means that the argmax of $\sup_{v\in F(x)} \langle v,p\rangle$ is unique - we denote it by $f_p(x) \in F(x)$.

We have also assumed $x \mapsto f_p(x)$ is Lipschitz . If $p(\cdot)$ is any continuous function, then the ODE

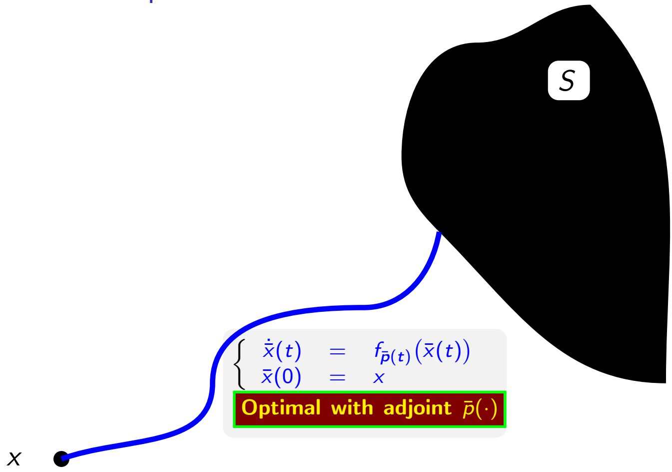
$$\textbf{(ODE)}_x \qquad \begin{cases} \dot{x}(t) = f_{p(t)}\big(x(t)\big) & \text{a.e. } t \in [0,T] \\ x(0) = x, \end{cases}$$

satisfies standard Carathéodory-type assumptions, and has the properties:

- A unique solution $x(\cdot;x)$ of $(\mathrm{ODE})_x$ exists on $[0,\infty)$;
- Each solution $x(\cdot;x)$ is a solution of **(DI)**;
- The function $x \mapsto x(t;x)$ is locally Lipschitz;
- If $\big(\bar{x}(\cdot),\bar{p}(\cdot)\big)$ is a Hamiltonian arc, then $x(\cdot)$ satisfies **(ODE)** with $p(\cdot) = \bar{p}(\cdot)$.

A replacement for a priori estimates

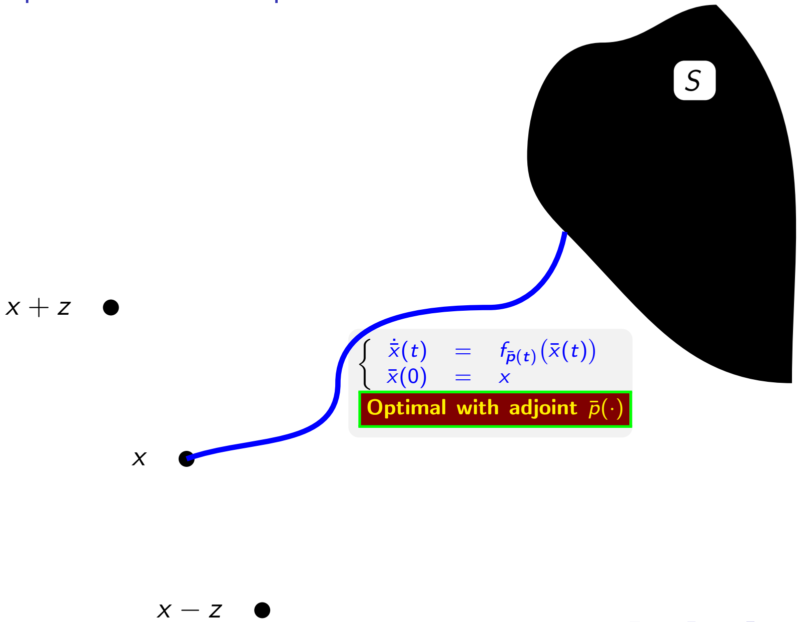
A replacement for a priori estimates



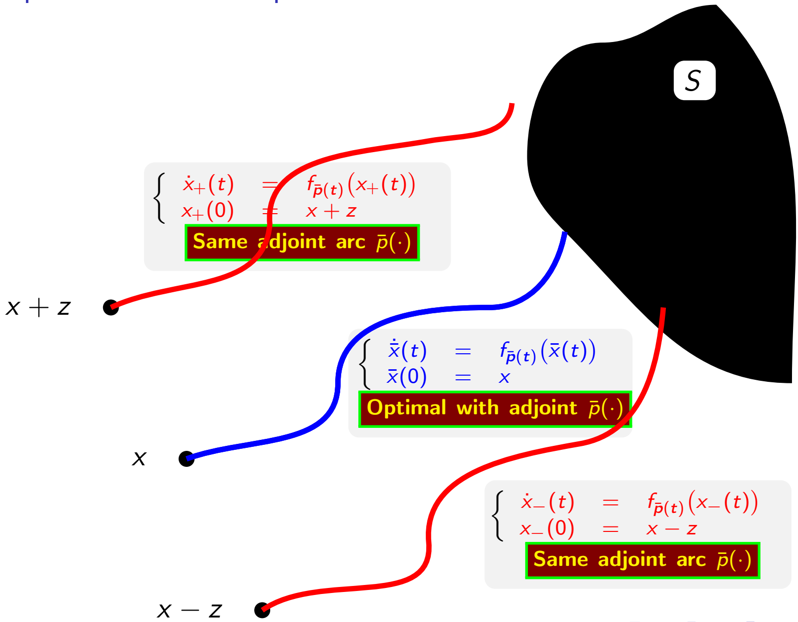
$$\begin{cases} \dot{\bar{x}}(t) & = & f_{\bar{p}(t)}\big(\bar{x}(t)\big) \\ \bar{x}(0) & = & x \end{cases}$$

Optimal with adjoint $\bar{p}(\cdot)$

A replacement for a priori estimates



A replacement for a priori estimates



Summary of new results

We assume F satisfies $(\mathrm{SH})_+$ and (H).

Summary of new results

We assume F satisfies (SH)$_+$ and (H).

Theorem

Consider the minimum time problem. Suppose the target S is compact and satisfies the Petrov condition and the Interior Sphere Property. Then there exists $\rho > 0$ so that $T(\cdot)$ is semiconcave on each convex subset of $S + \rho\overline{\mathbb{B}} \setminus S$.

Summary of new results

We assume F satisfies (SH)$_+$ and (H).

Theorem

Consider the minimum time problem. Suppose the target S is compact and satisfies the Petrov condition and the Interior Sphere Property. Then there exists $\rho > 0$ so that $T(\cdot)$ is semiconcave on each convex subset of $S + \rho\overline{\mathbb{B}} \setminus S$.

Theorem

Consider the Mayer problem. Assume the endpoint cost function $\ell(\cdot)$ is semiconcave. Then the value function $V(\cdot, \cdot)$ is locally semiconcave on $(-\infty, T] \times \mathbb{R}^n$.

Conclusions and future work

- **General Philosophy: Assumptions should not be required only in the proof.** Here, smooth parameterizations of control systems should not be assumed if the conclusion does not reflect them.

Conclusions and future work

- **General Philosophy: Assumptions should not be required only in the proof.** Here, smooth parameterizations of control systems should not be assumed if the conclusion does not reflect them.
- We offered an alternative; It is not yet clear how generic these assumptions are. For example, could they be invoked in an auxiliary problem that approximates an original one?

Conclusions and future work

- **General Philosophy: Assumptions should not be required only in the proof.** Here, smooth parameterizations of control systems should not be assumed if the conclusion does not reflect them.
- We offered an alternative; It is not yet clear how generic these assumptions are. For example, could they be invoked in an auxiliary problem that approximates an original one?
- Local versions?

Conclusions and future work

- **General Philosophy: Assumptions should not be required only in the proof.** Here, smooth parameterizations of control systems should not be assumed if the conclusion does not reflect them.
- We offered an alternative; It is not yet clear how generic these assumptions are. For example, could they be invoked in an auxiliary problem that approximates an original one?
- Local versions?
- Perhaps a large segment of optimal control can be cast in this way.(?)

Conclusions and future work

- **General Philosophy: Assumptions should not be required only in the proof.** Here, smooth parameterizations of control systems should not be assumed if the conclusion does not reflect them.
- We offered an alternative; It is not yet clear how generic these assumptions are. For example, could they be invoked in an auxiliary problem that approximates an original one?
- Local versions?
- Perhaps a large segment of optimal control can be cast in this way.(?)
- **Students: you're urged and invited to join the fun.**

Conclusions and future work

- **General Philosophy: Assumptions should not be required only in the proof.** Here, smooth parameterizations of control systems should not be assumed if the conclusion does not reflect them.
- We offered an alternative; It is not yet clear how generic these assumptions are. For example, could they be invoked in an auxiliary problem that approximates an original one?
- Local versions?
- Perhaps a large segment of optimal control can be cast in this way.(?)
- **Students: you're urged and invited to join the fun.**

Thank you for your attention and for sticking around.

Thank you for your attention!

Grazie per l'attenzione!

Dziękuję za uwagę!

Merci de l'attention

Obrigado pela atenção

Vielen Dank für Ihre Aufmerksamkeit

Gracias por su atención

Gràcies per la vostra atenció

Cám on

شكرا لاهتمامكم

And finally ...

And finally ...

Many thanks to Richard, Rosario, Hasnaa, Estelle, and all the SADCO people!

And finally ...

Many thanks to Richard, Rosario, Hasnaa, Estelle, and all the SADCO people!

It's been a great week!